極端降雨空間變異分析模擬與應用

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Outline

- **Introduction**
- **Improving accuracy of frequency analysis**
  - Presence of outliers (extraordinary rainfall extremes)
  - Spatial correlation (non-Gaussian random field simulation)
- **Frequency analysis of multi-site rainfall extremes**

Introduction

- **Frequent occurrences of disasters induced by heavy rainfalls in Taiwan**
  - Landslides
  - Debris flows
  - Flooding and urban inundation
- **Increasing occurrences of rainfall extremes (some of them are record breaking) in recent years**
- **Almost all of the long-duration rainfall extremes (longer than 12 hours) were produced by typhoons.**

Examples of annual max events in Taiwan

Design durations

1965-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/5	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18
Wutuh	6/11	8/21	9/6	9/6	9/6	9/6	8/18	8/18	8/18	8/18
1969-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/14	5/14	5/14	5/14	9/26	9/26	9/9	9/9	9/9	9/9
Wutuh	9/21	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8
1974-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/27	9/27	10/11	10/11	10/11	10/11	10/11	10/11	10/11	10/11
Wutuh	9/15	9/15	10/11	10/11	9/15	10/11	10/11	10/11	10/11	10/11

1983-year

Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/23	5/23	5/23	5/23	5/23	10/10	10/10	10/10	10/10	10/10
Wutuh	10/1	10/1	6/3	6/3	6/3	10/1	10/1	10/1	10/1	10/1

1987-year

Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22
Wutuh	10/22	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22

1994-year

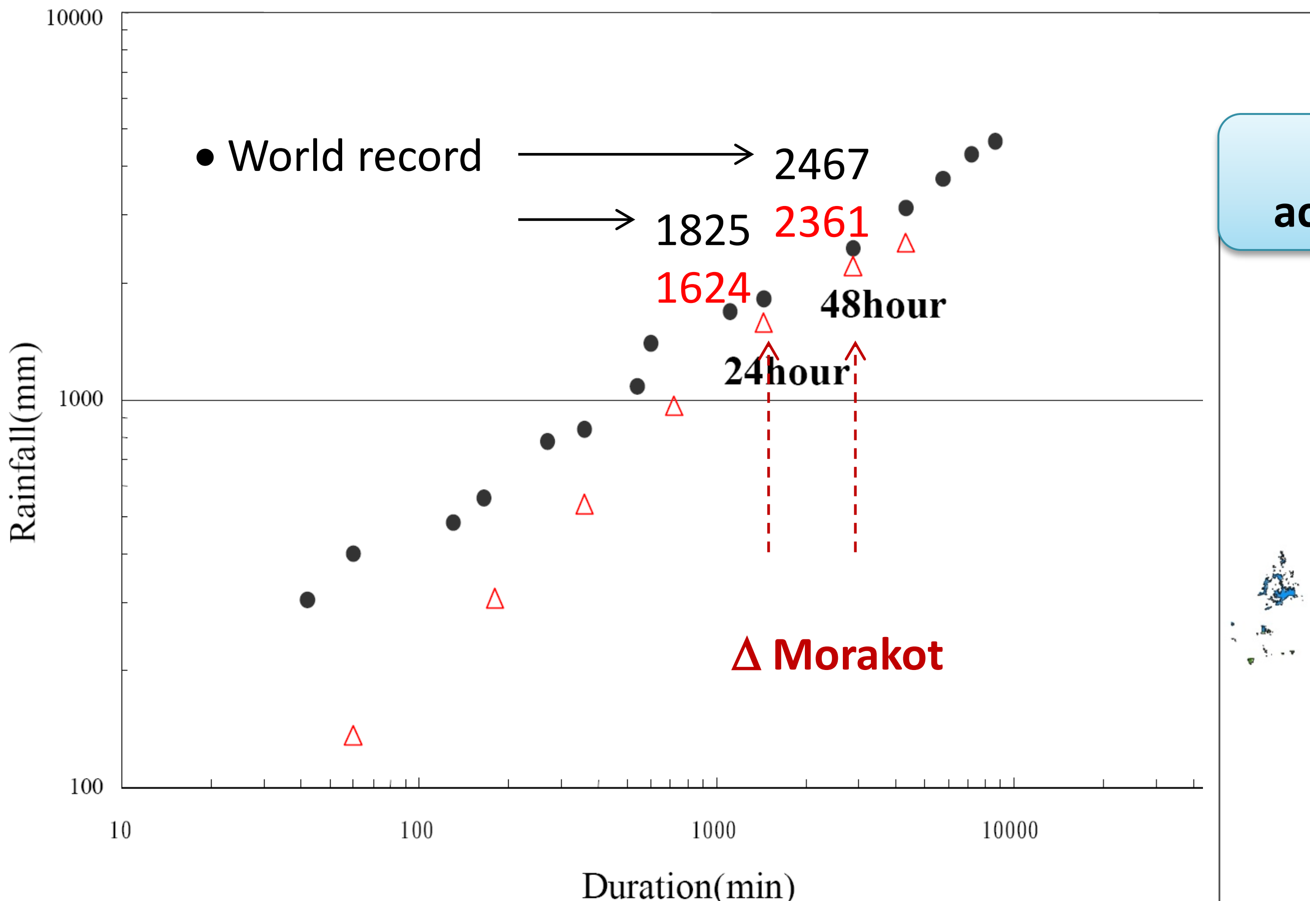
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	6/18	6/18	6/18	6/18	6/18	10/9	10/9	10/9	10/9	10/9
Wutuh	6/18	6/18	6/18	9/12	9/12	9/12	9/12	9/12	9/12	9/12

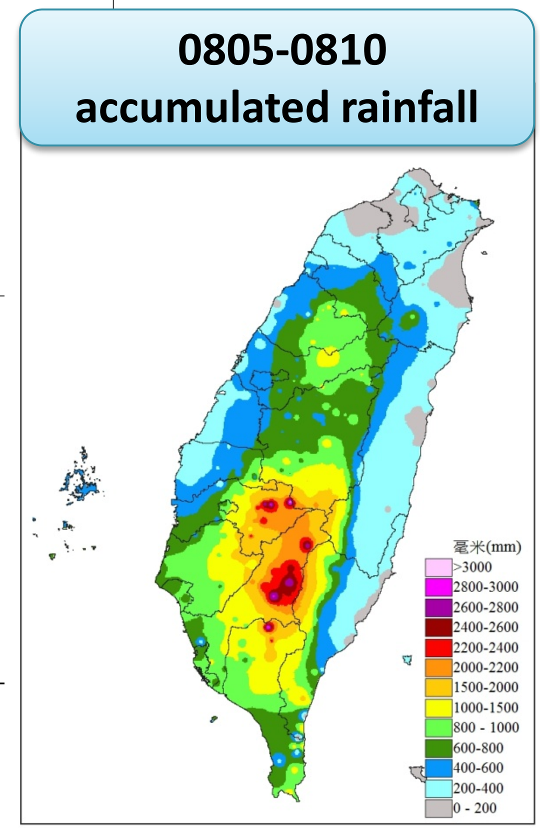
19840624	WYNNE魏恩	26	7	0.269231
19850626	-	28	6	0.214286
19860919	ABBY艾貝	28	11	0.392857
19870727	ALEX亞力士	28	11	0.392857
19880813	-	18	10	0.555556
19890912	SARAH莎拉	25	20	0.8
19900623	OFELIA歐菲莉	28	16	0.571429
19910622	-	26	8	0.307692
19920830	POLLY寶莉	27	18	0.666667
19930912	ABE亞伯	20	10	0.5
19940803	CAITLIN凱特琳	28	19	0.678571
19950608	DEANNA薇安娜	28	13	0.464286
19960731	HERB賀伯	28	21	0.75
19970828	AMBER安珀	26	13	0.5
19980804	OTTO奧托	27	14	0.518519
19990811	-	27	6	0.222222
20000822	BILIS碧利斯	27	17	0.62963
20010917	NARI納莉	27	13	0.481481
20020805	-	28	12	0.428571
20030606	-	28	13	0.464286
20040702	MINDULLE敏督利	28	24	0.857143
20050718	HAITANG海棠	28	17	0.607143
20060609	-	28	19	0.678571
20071006	KROSA柯羅莎	28	9	0.321429
20080717	KALMAEGI 卡玫基	28	18	0.642857
20090808	MORAKOT 莫拉克	27	26	0.962963
20100919	FANAPI 凡那比	27	18	0.666667

Proportion of rainfall stations observed annual max rainfalls during the event.

- **A large number of rain gauges maintained by CWB and WRA. Some of them have record length longer than 50 years. Most of them have less than 30 years record length .**
- **Design rainfalls play a key role in studies related to climate change and disaster mitigation.**
- **Problems in rainfall frequency analysis**
  - Short record length (less than 30 years) (**small sample size**)
  - Record breaking rainfall extremes (**presence of extreme outliers**)
    - Typhoon Morakot

• Typhoon Morakot (2009)



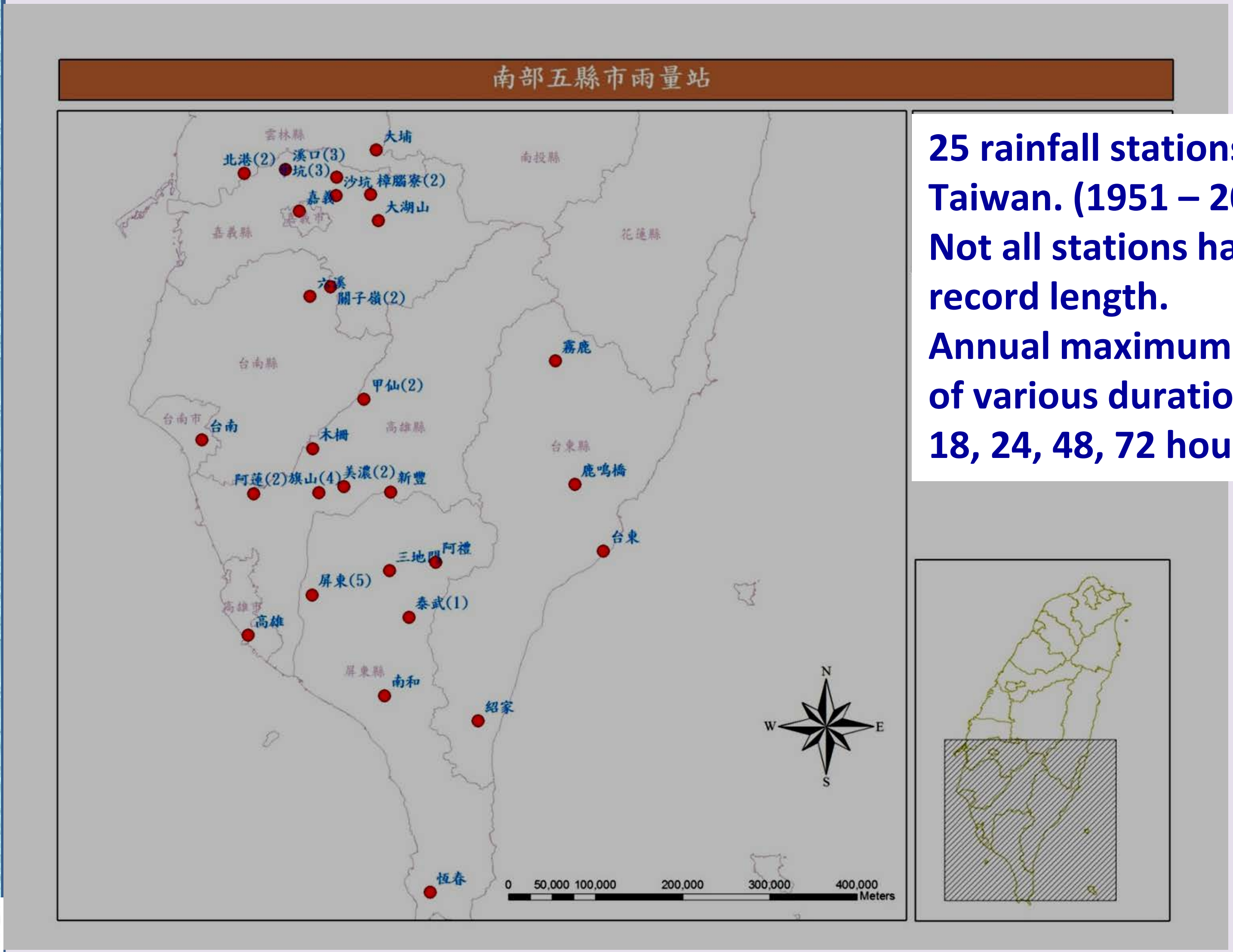


- Catastrophic storm rainfalls (or extraordinary rainfalls) often are considered as extreme outliers. Whether or not such rainfalls should be included in site-specific frequency analysis is disputable.
  - 24-hr annual max. rainfalls (Morakot) of 2009
    - 甲仙 1077 mm
    - 泰武 1747 mm
    - 大湖山 1329 mm
    - 阿禮 1237 mm

- **Concurrent occurrences of extraordinary rainfalls at different rain gauges**
  - Several stations had 24-hr rainfalls exceeding 100-yr return period.
  - Site-specific events of 100-yr return period.
  - What is the return period of the event of multi-site 100-yr return period?
    - (100)^4 = 100,000,000 years (4 sites), assuming independence
  - **Redefining extreme events**
    - **multi-site extreme events** w.r.t. specified durations and rainfall thresholds
    - Spatial covariation of rainfall extremes

- **Spatial covariation of rainfall extremes**
  - By using site-specific annual maximum rainfall series for frequency analysis implies a significant loss of valuable information.

Study area and rainfall stations



25 rainfall stations in southern Taiwan. (1951 – 2010)
Not all stations have the same record length.
Annual maximum rainfalls (AMR) of various durations (1, 2, 6, 12, 18, 24, 48, 72 hours)

Event-maximum rainfalls of various design durations

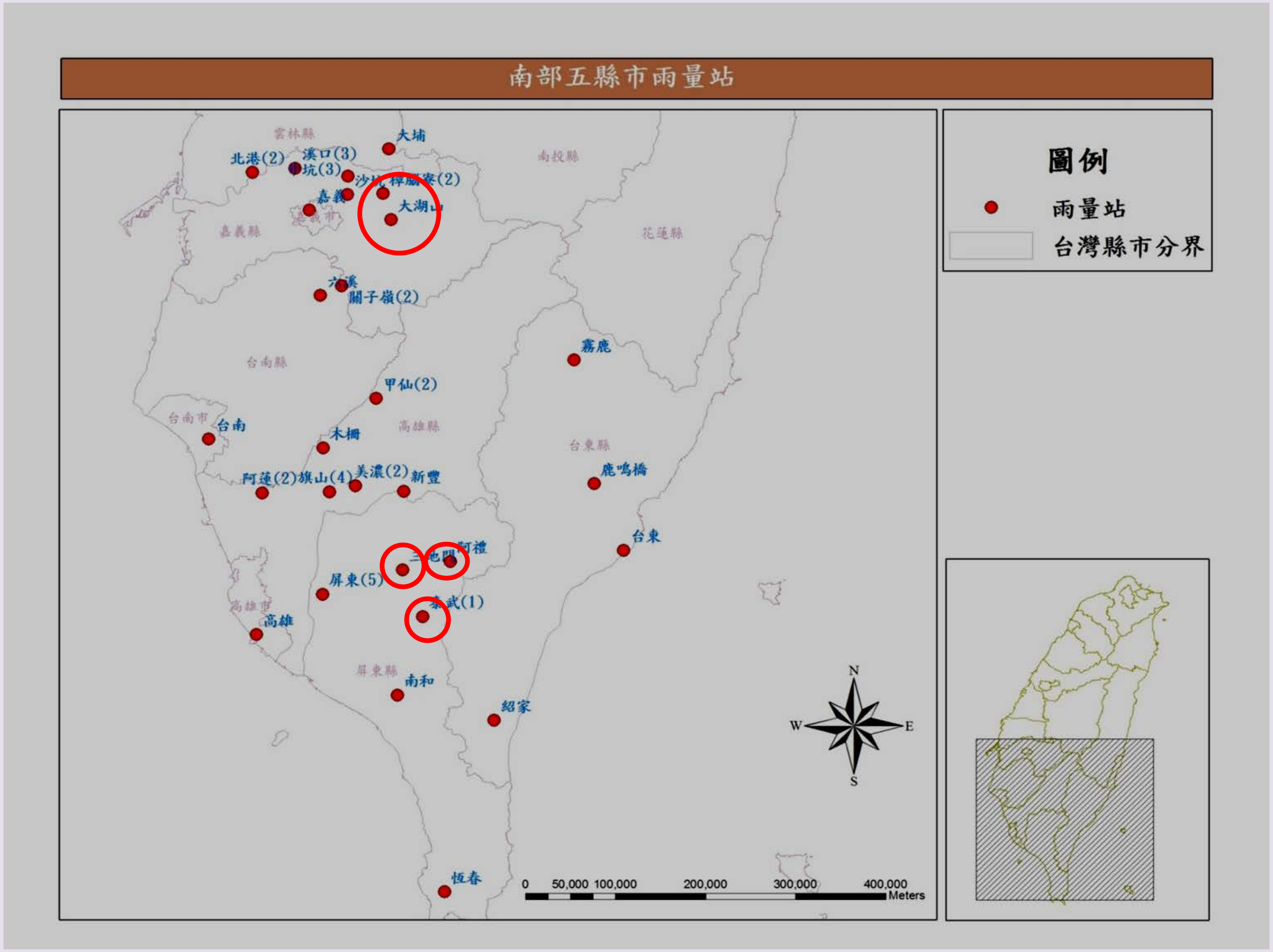
- **Event-max 1, 2, 6, 12, 18, 24, 48, 72-hr typhoon rainfalls at individual sites.**
- **Approximately 120 events**
- **Complete series**

Delineation of homogeneous regions (K-mean cluster analysis)

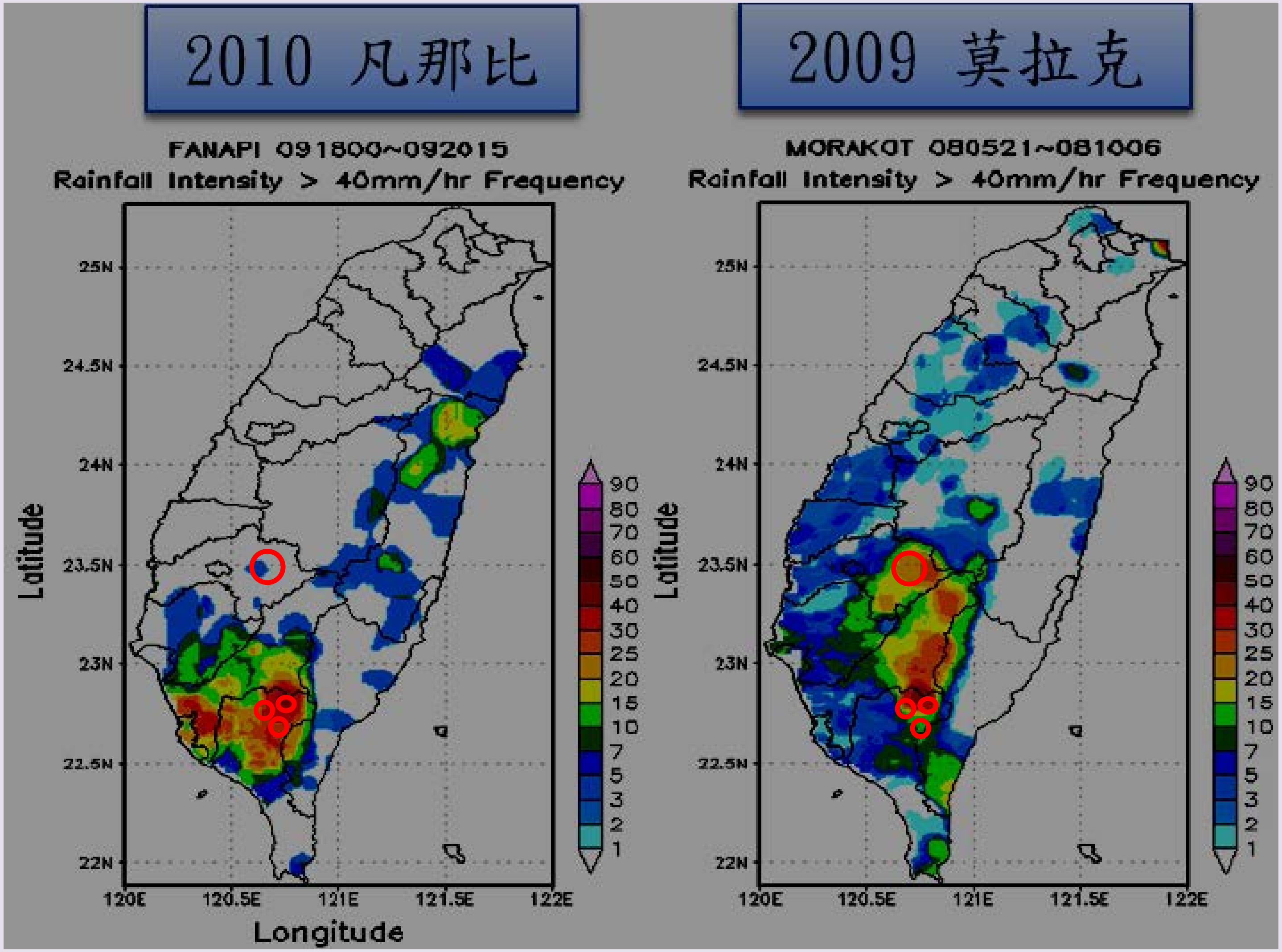
- **24 classification features (8 design durations x 3 parameters – mean, std dev, skewness)**
  - Normalization of individual features
- **Two homogeneous regions (25 stations)**

Regional frequency analysis

- **Delineating homogeneous regions**

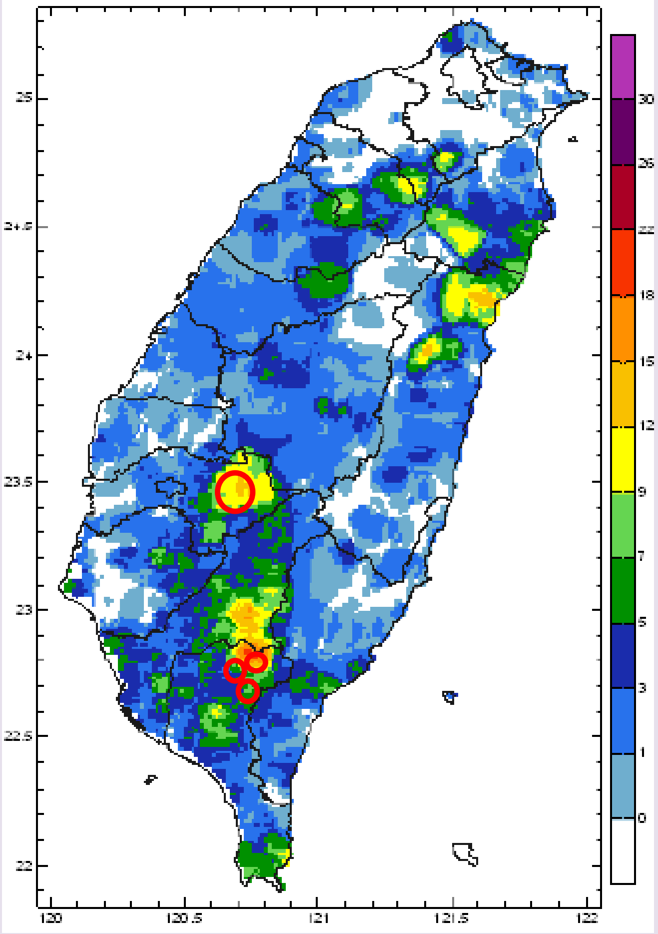


颱風強降雨頻率分布



2010 凡那比
FANAPI 091800~092015
Rainfall Intensity > 40mm/hr Frequency
2009 莫拉克
MORAKOT 080521~081006
Rainfall Intensity > 40mm/hr Frequency
25N
24.5N
24N
23.5N
23N
22.5N
22N
120E
120.5E
121E
121.5E
122E
Latitude
Longitude
90
80
70
60
50
40
30
25
20
15
10
7
5
3
2
1

Hot spots for occurrences of extreme rainfalls

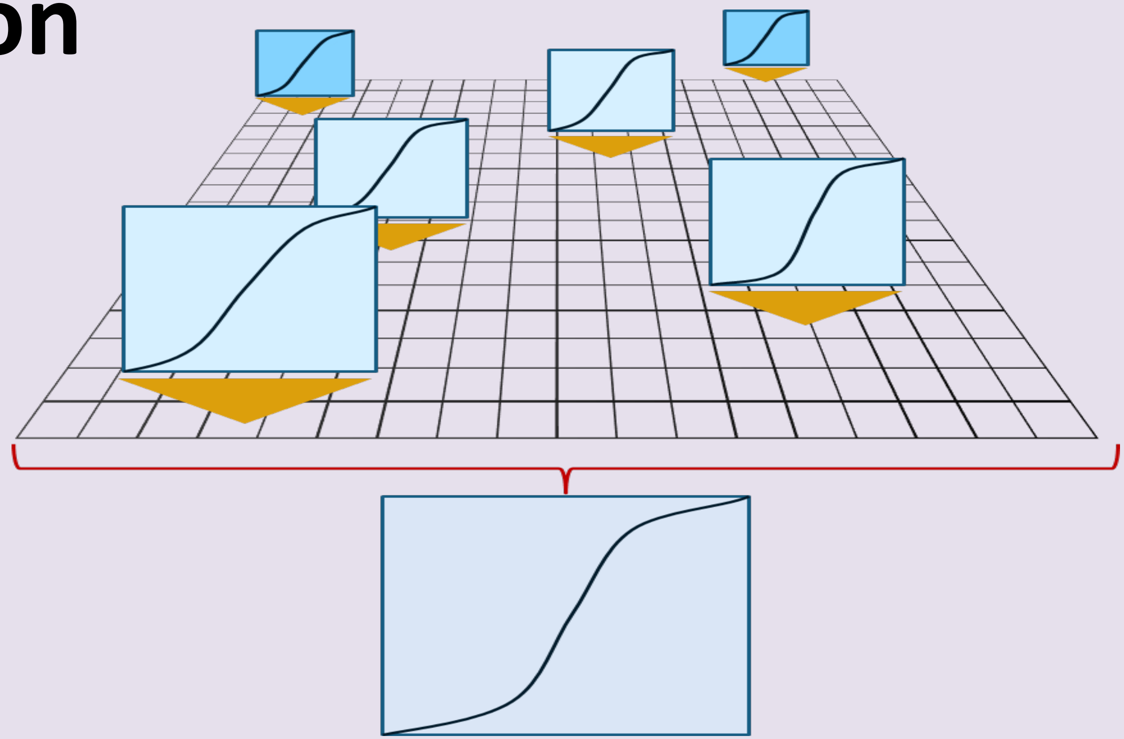


1992 – 2010
Number of extreme typhoon events

Rescaled variables for regional frequency analysis – Frequency factors

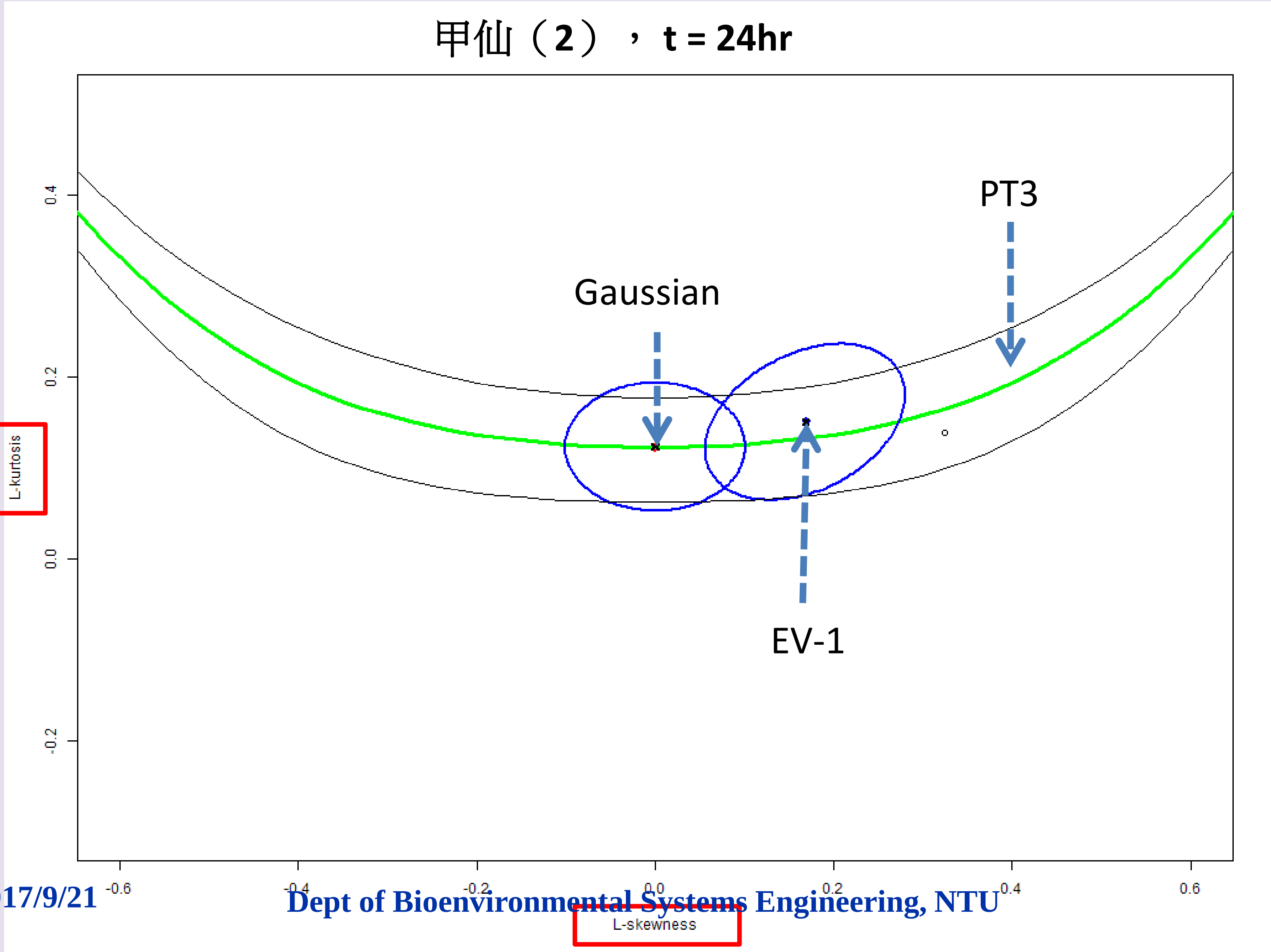
$$K_{ijk} = \frac{x_{ijk} - \mu_{ik}}{\sigma_{ik}}$$

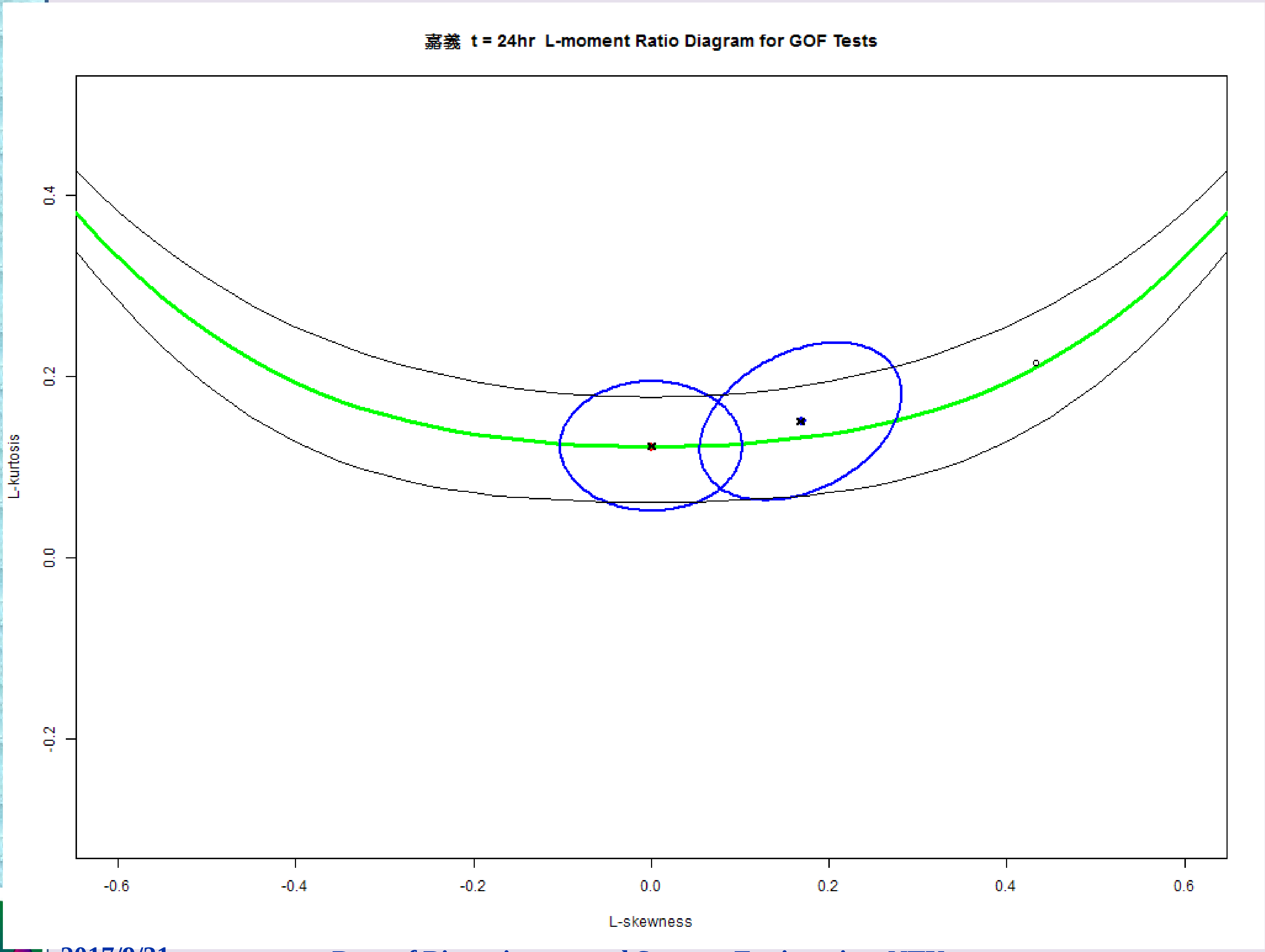
Goodness-of-fit test and parameters estimation



- L-moment ratio diagrams (LMRD) for goodness-of-fit test
  - Pearson Type 3 distribution
- Parameters estimation
  - Method of L-moments
  - Record-length-weighted regional parameters

L-moment-ratio diagram GOF test




嘉義 t = 24hr L-moment Ratio Diagram for GOF Tests
L-kurtosis
-0.2
0.0
0.2
0.4
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
L-skewness

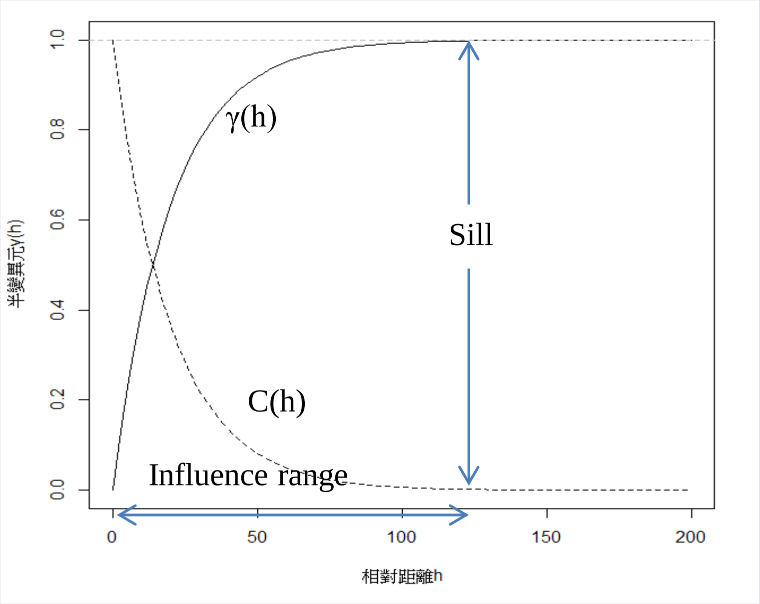
Covariance structure of the random field

- Covariance matrices are semi-positive definite.
- Experimental covariance matrices often do not satisfy the semi-positive definite condition.
- Modeling the covariance structure of frequency factors (event-max rainfalls) by variogram modeling.

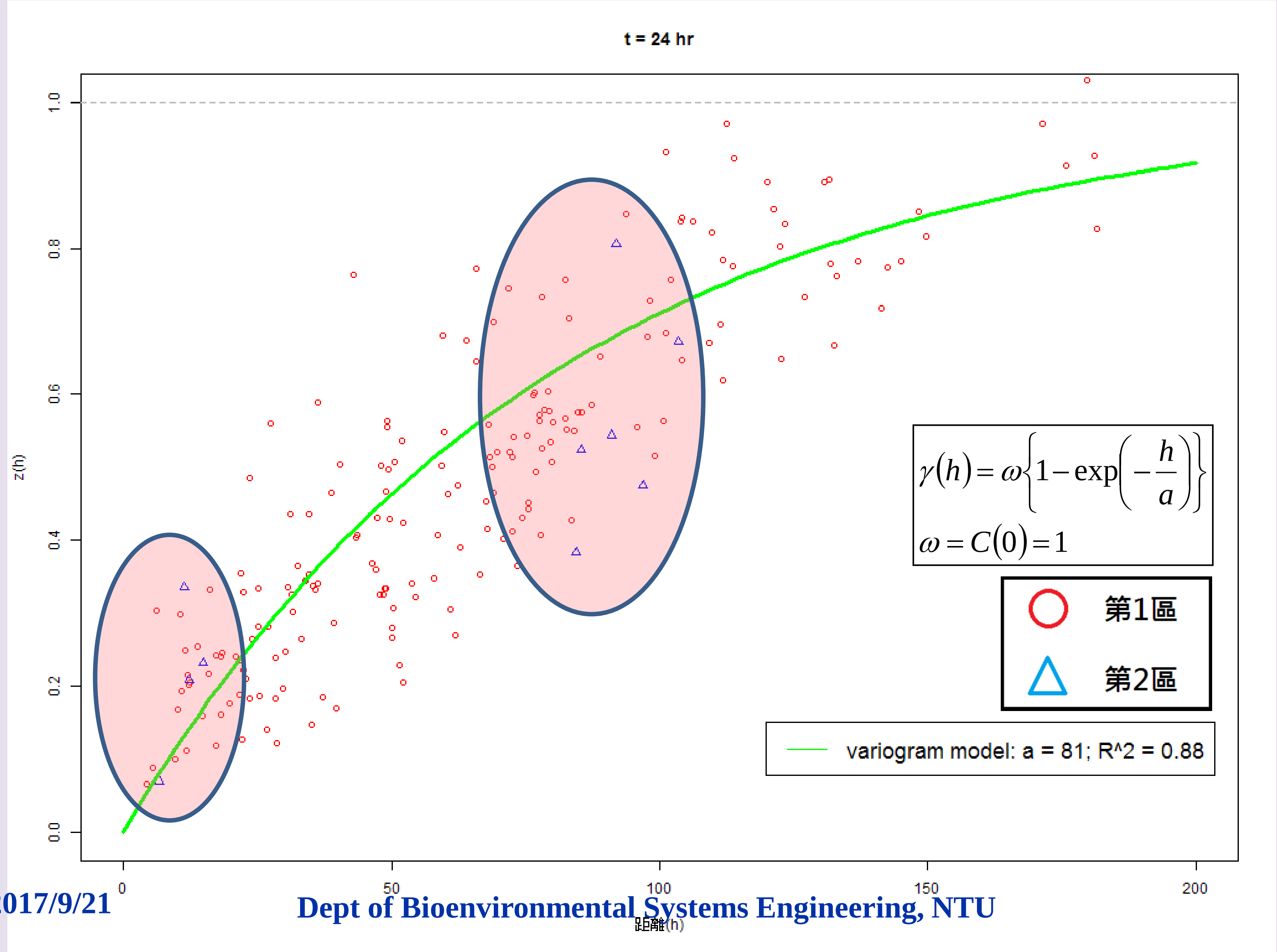
Relationship between semivariogram and covariance function

$$\gamma(h) = \frac{1}{2} Var[Z(x+h) - Z(x)]$$
$$= C(0) - C(h)$$

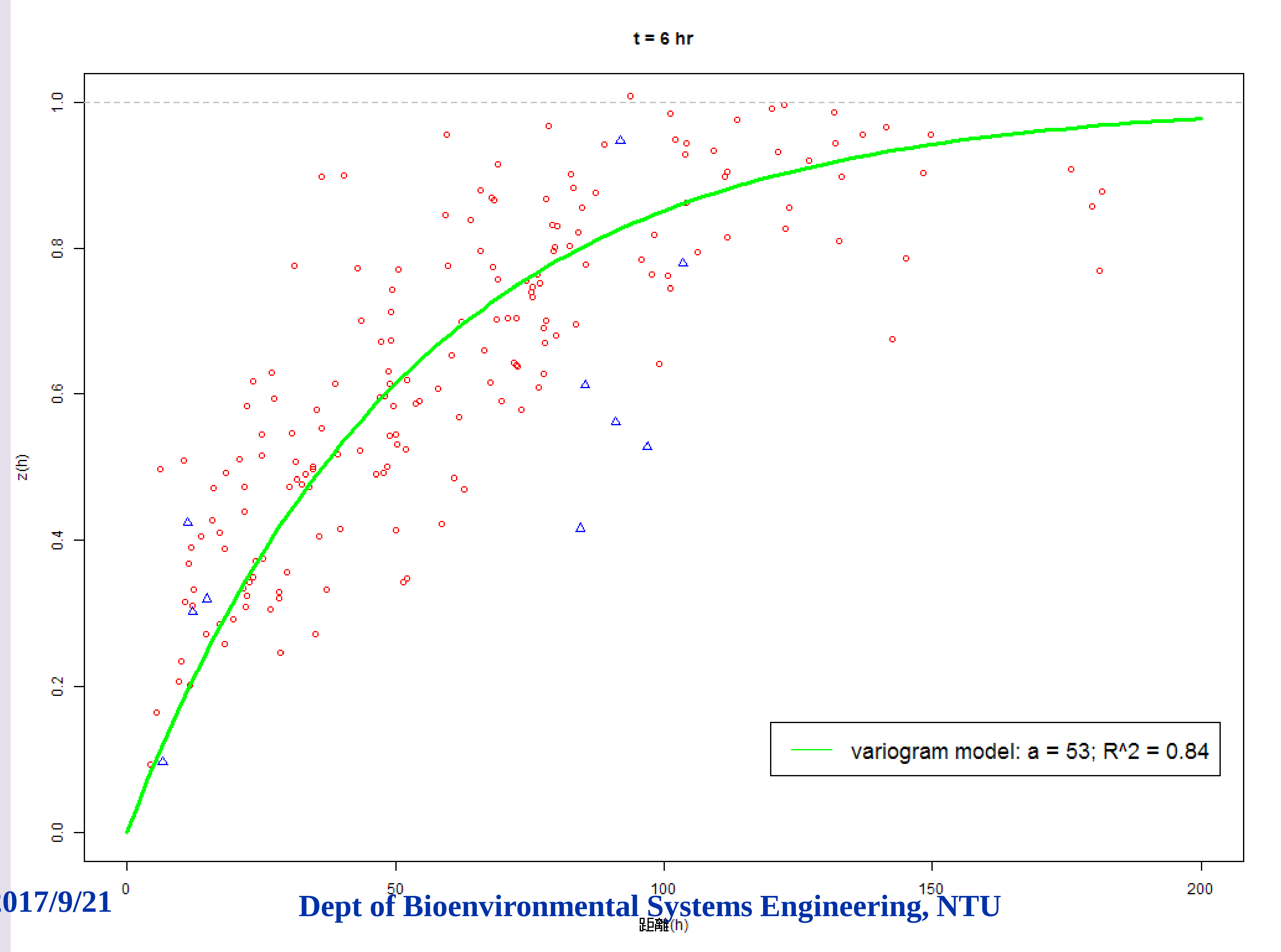
$$C(x_i, x_j) = C(h)$$
$$sill = C(0) = 1$$



Semi-variogram modeling (24-hr EMR)



Semi-variograms of EMRs of other durations



Stochastic Simulation of Multi-site Event-Max Rainfalls

- **Pearson type III (Non-Gaussian) random field simulation**
- **Covariance Transformation Approach**
  - Covariance matrix of multivariate PT3 distribution
  - Covariance matrix of multivariate standard Gaussian distribution
  - Multivariate Gaussian simulation
  - Transforming simulated multivariate Gaussian realizations to PT3 realizations

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV} + 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$

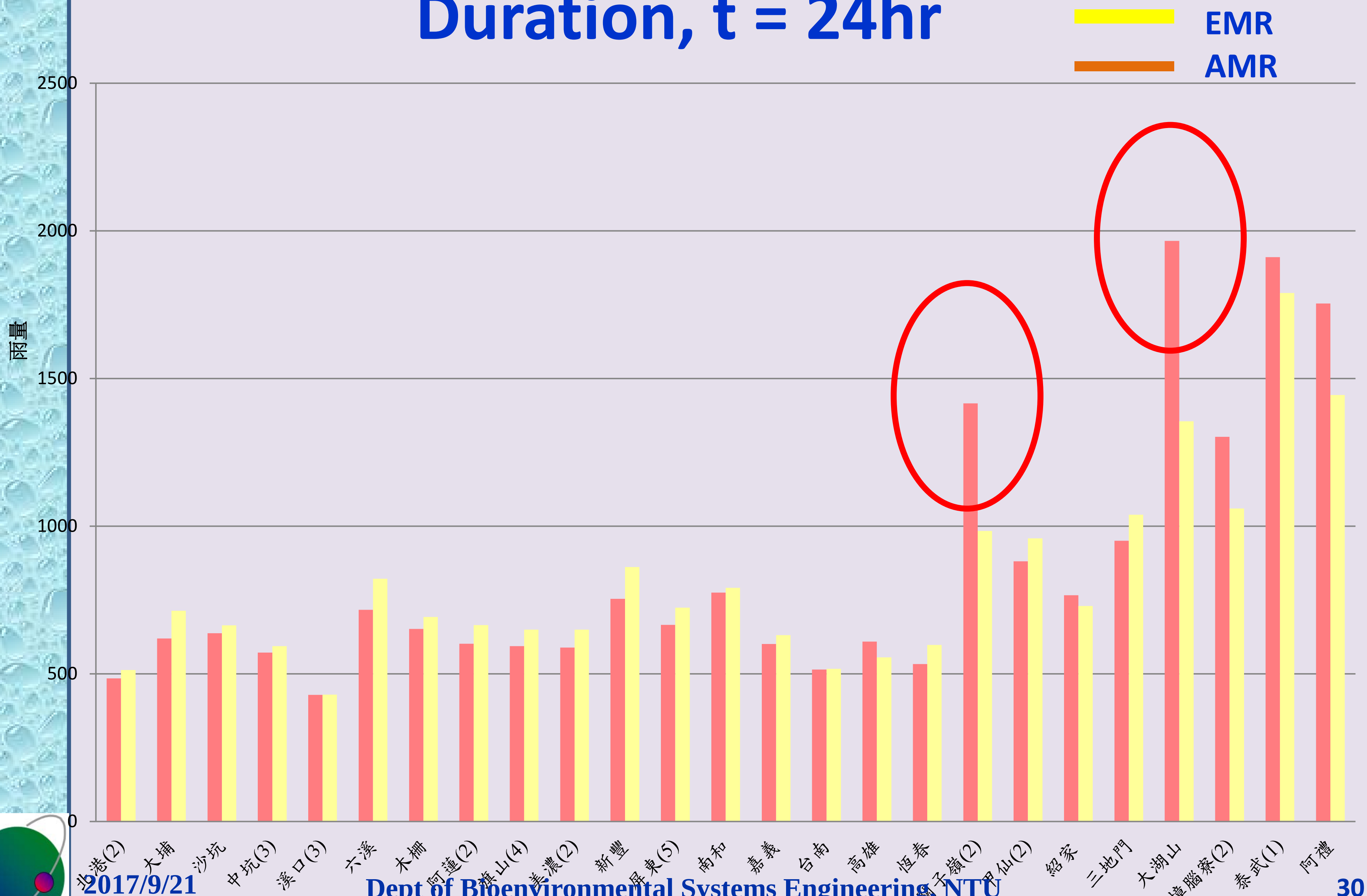
$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \quad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \quad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \quad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \quad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

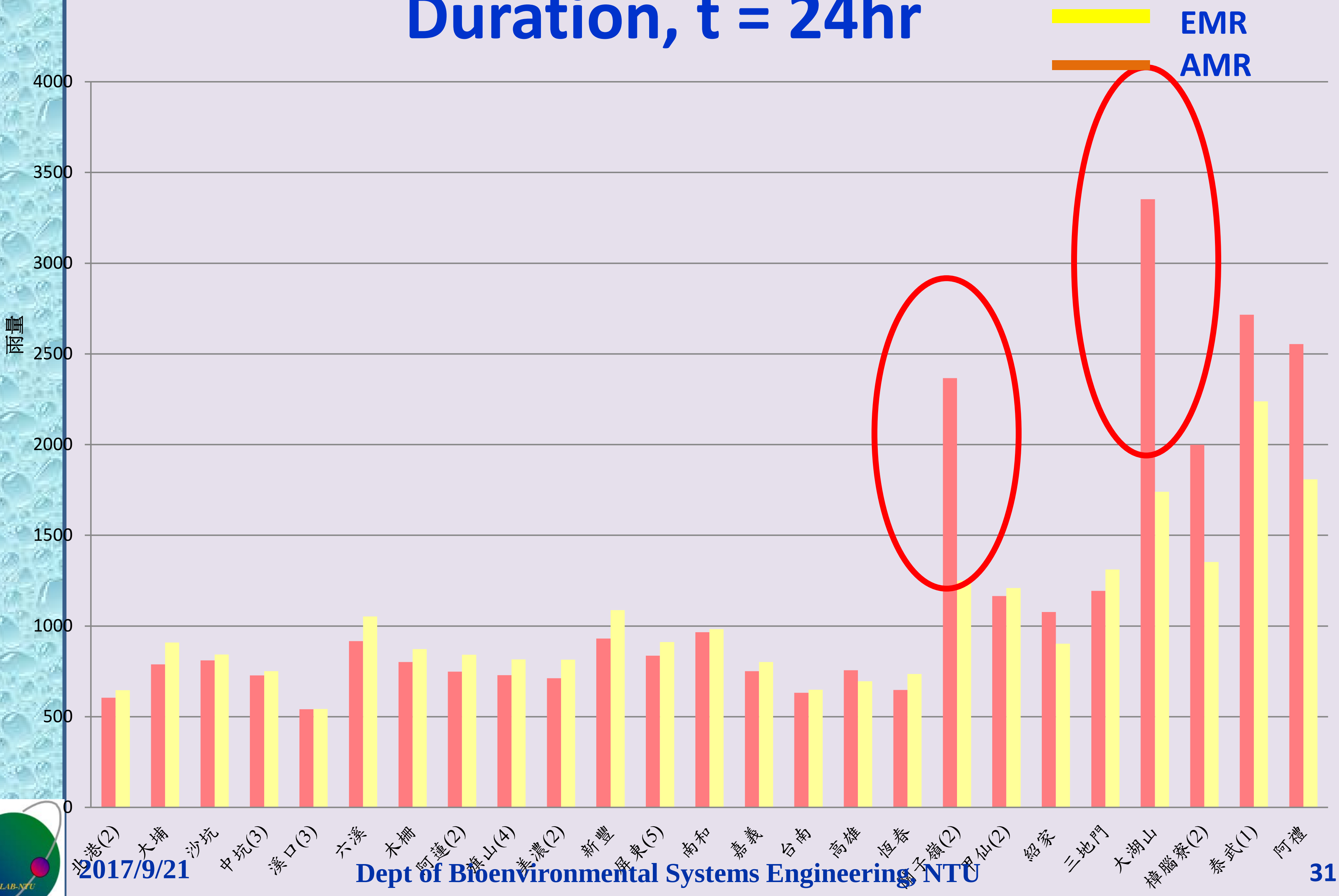
- **Each simulation run generated one sample of t-hr multi-site event maximum rainfalls.**
- **Simulated samples preserved the spatial covariation of multi-site EMRs as well as the marginal distributions.**
- **10,000 samples were generated.**
  - Multi-site t-hr EMRs of 10,000 typhoon events.
- **The number of typhoons vary from one year to another.**
  - **Annual count of typhoons is a random variable.**

- **Determination of t-hr rainfall of T-yr return period**
  - Average number of typhoons per year, m = 2.43
  - Return period, T=100 years
  - Exceedance probability of the event-max rainfalls, $p_E=1/(100*2.43)$

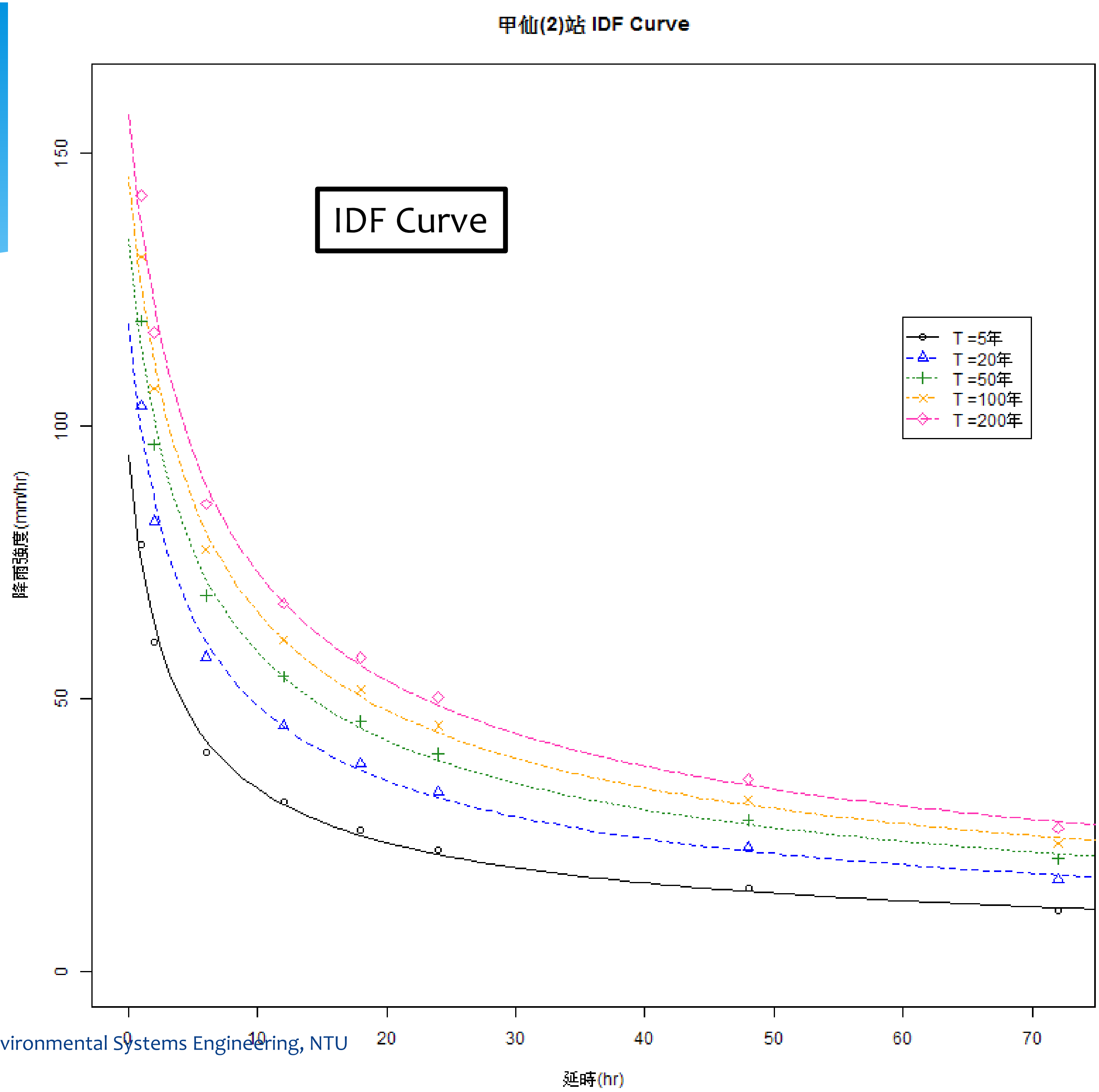
Return period, T = 50yr
Duration, t = 24hr



Return Period, T = 200yr
Duration, t = 24hr



	24hr雨量(mm)	24hr重現期(年)	48hr雨量(mm)	48hr重現期(年)	72hr雨量(mm)	72hr重現期(年)
北港(2)	358	11	433	11	460	12
大埔	720	53	921	57	1069	118
沙坑	607	32	759	31	888	52
中坑(3)	497	24	636	26	744	42
溪口(3)	324	15	403	16	450	24
六溪	791	43	1024	55	1122	59
木柵	778	102	1037	54	1188	80
阿蓮(2)	630	41	868	42	924	41
旗山(4)	519	17	760	23	819	22
美濃(2)	338	4	509	6	569	6
新豐	908	63	1179	51	1195	36
屏東(5)	676	40	878	45	938	45
南和	614	15	988	39	1100	62
嘉義	526	22	646	17	697	24
台南	535	61	709	53	736	42
高雄	538	40	755	57	801	53
恆春	528	27	715	34	730	28
關子嶺(2)	1098	92	1490	126	1683	221
甲仙(2)	1040	65	1614	177	1915	204
紹家	818	110	1176	141	1333	130
三地門	825	18	1109	25	1235	29
大湖山	1329	48	1958	89	2533	306
樟腦寮(2)	868	22	1395	55	1846	141
泰武(1)	1747	47	2938	121	3417	171
阿禮	1237	23	1908	38	2335	66

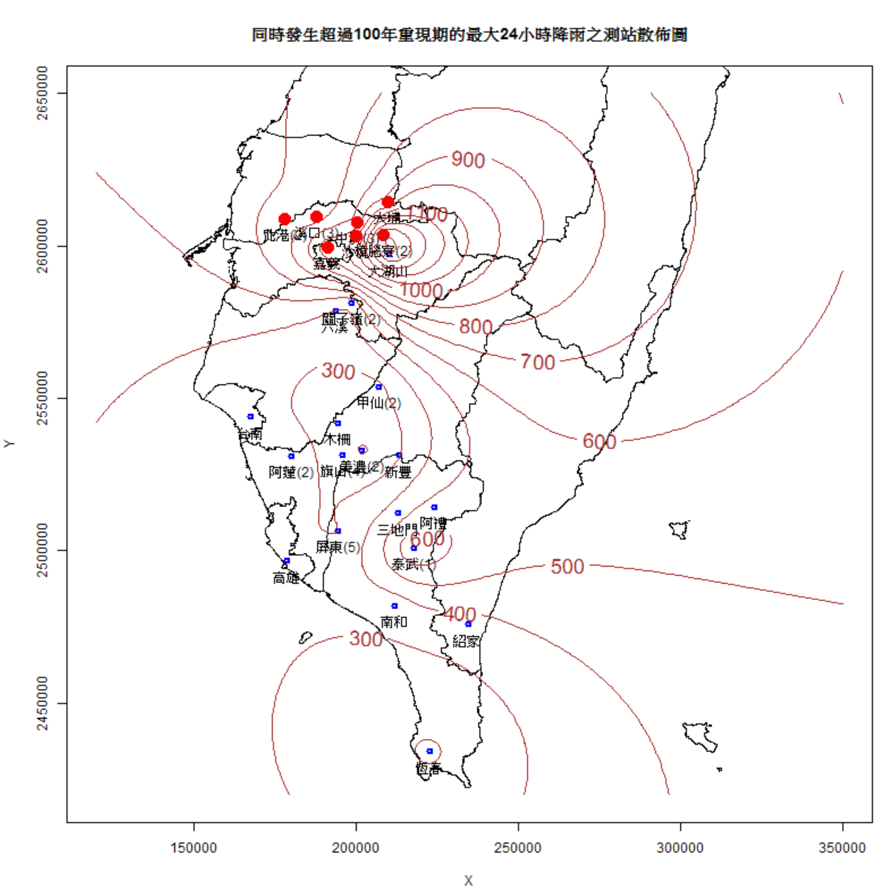

甲仙(2)站 IDF Curve
IDF Curve
T =5年
T =20年
T =50年
T =100年
T =200年
降雨強度(mm/hr)
延時(hr)
0
50
100
150
0
10
20
30
40
50
60
70

Estimating return period of multisite Morakot rainfall extremes

- **Four rain gauges within the Kaoping River Basin recorded 24-hr rainfalls close to 1,000mm (908, 1040, 825, 1237 mm).**
- **Define a multi-site extreme event**
  - over 1,000 mm 24-hr-rainfalls at all four sites
  - Among the 10,000 simulated events, only 8 events satisfied the above requirement.
  - Average number of typhoons per year, m=2.43
  - Multi-site extreme event return period T = 514 years.

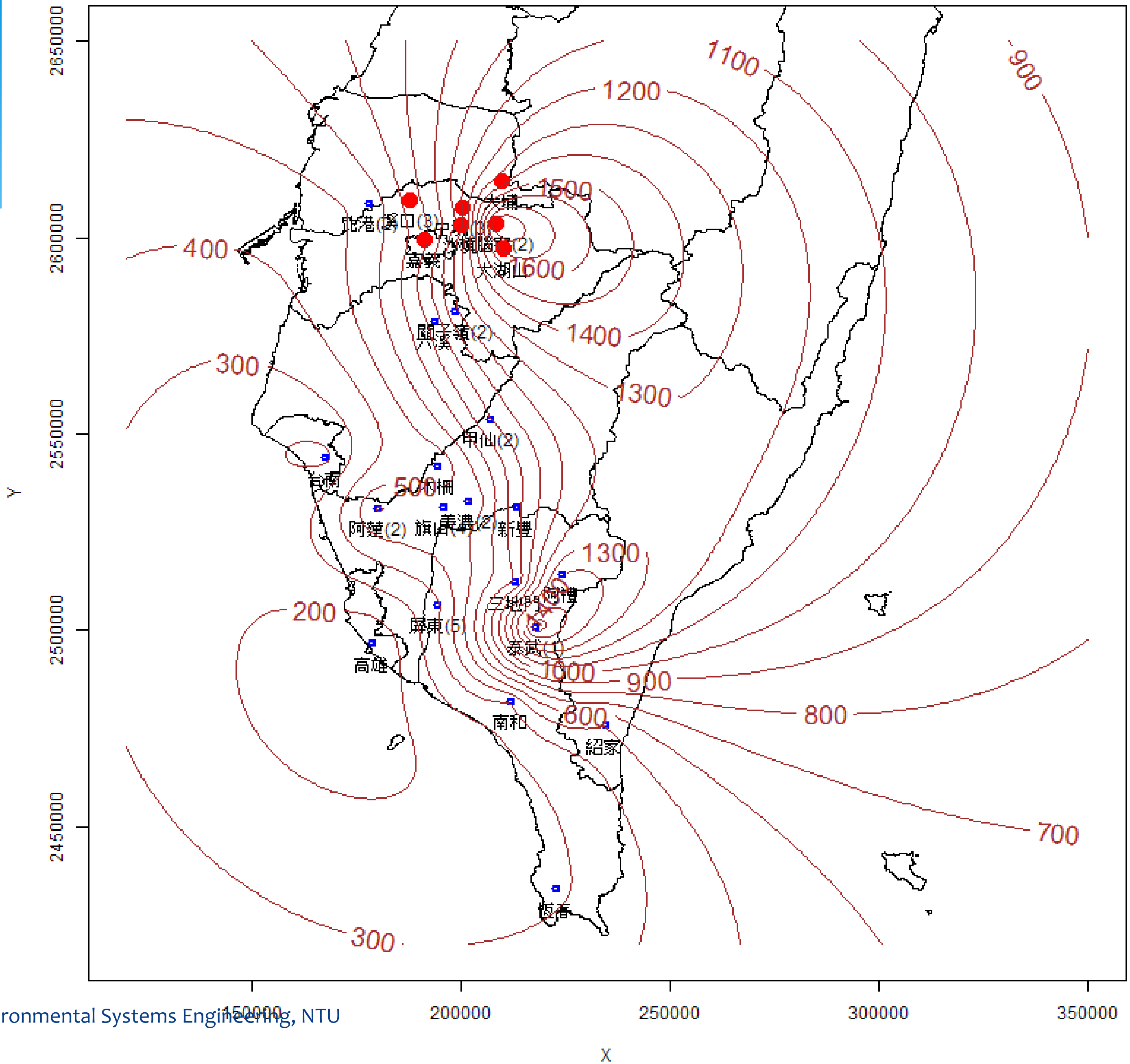
- **Further look at the simulation results**
  - Preserving the spatial pattern of rainfall extremes

Type A

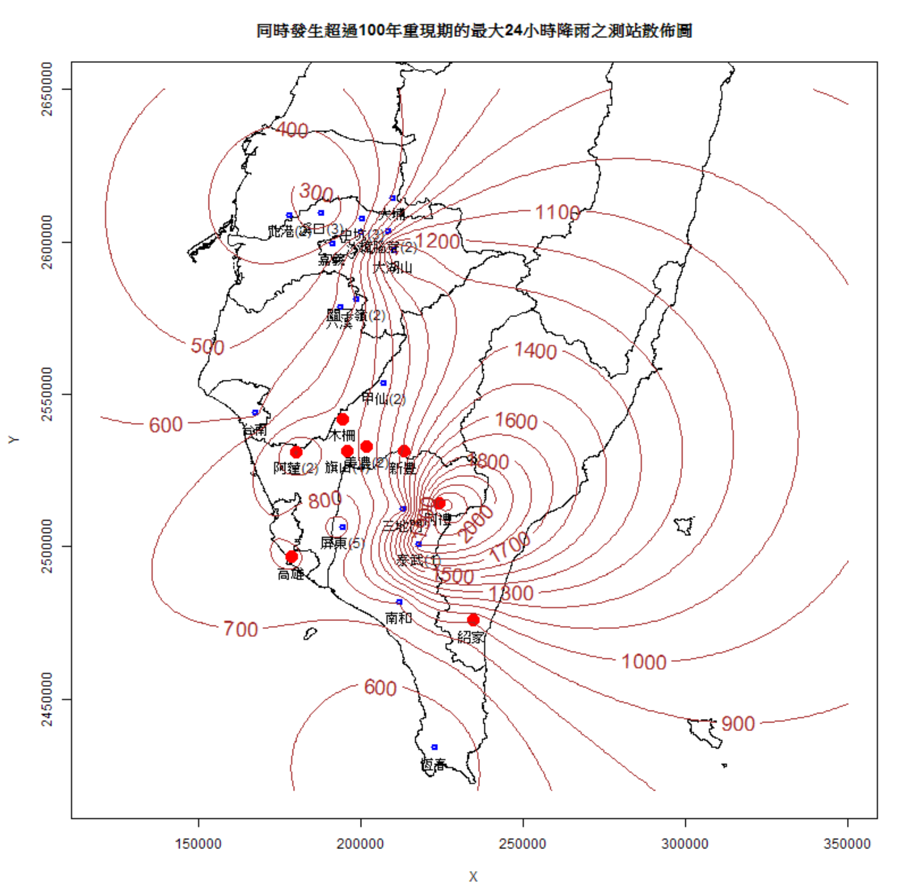


同時發生超過100年重現期的最大24小時降雨之測站散佈圖
900
1100
1000
800
700
600
500
400
300
甲仙(2)
木柵
台南
阿蓮(2)
新豐
三地門
阿禮
屏東(5)
高雄
南和
紹家
恆春
2650000
2600000
2550000
2500000
2450000
150000
200000
250000
300000
350000
X
Y

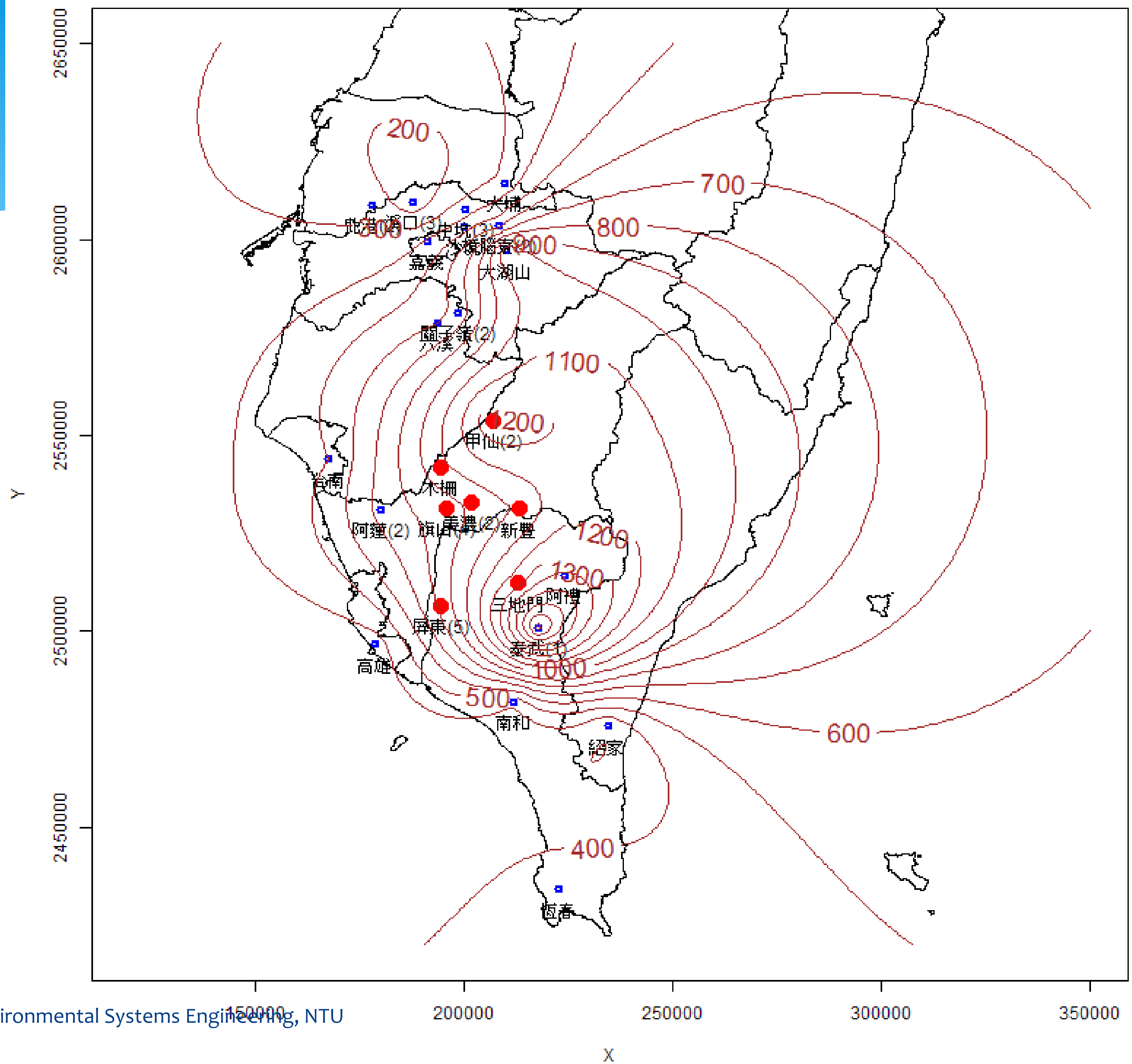
同時發生超過100年重現期的最大24小時降雨之測站散佈圖



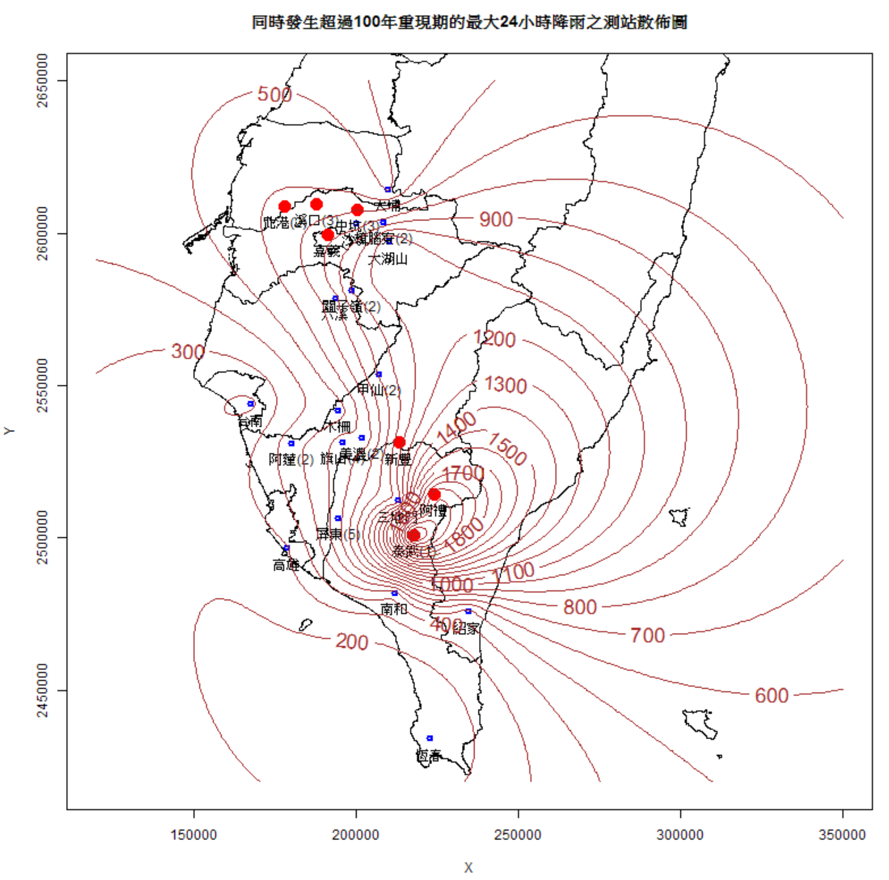
Type B


同時發生超過100年重現期的最大24小時降雨之測站數佈圖
2650000
2600000
2550000
2500000
2450000
Y
150000
200000
250000
300000
350000
X
400
300
1100
1200
500
1400
600
1600
1800
800
2000
1700
1500
1300
700
1000
600
900
大湖山
甲仙(2)
台南
木柵
新豐
屏東(5)
高雄
南和
紹家
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖

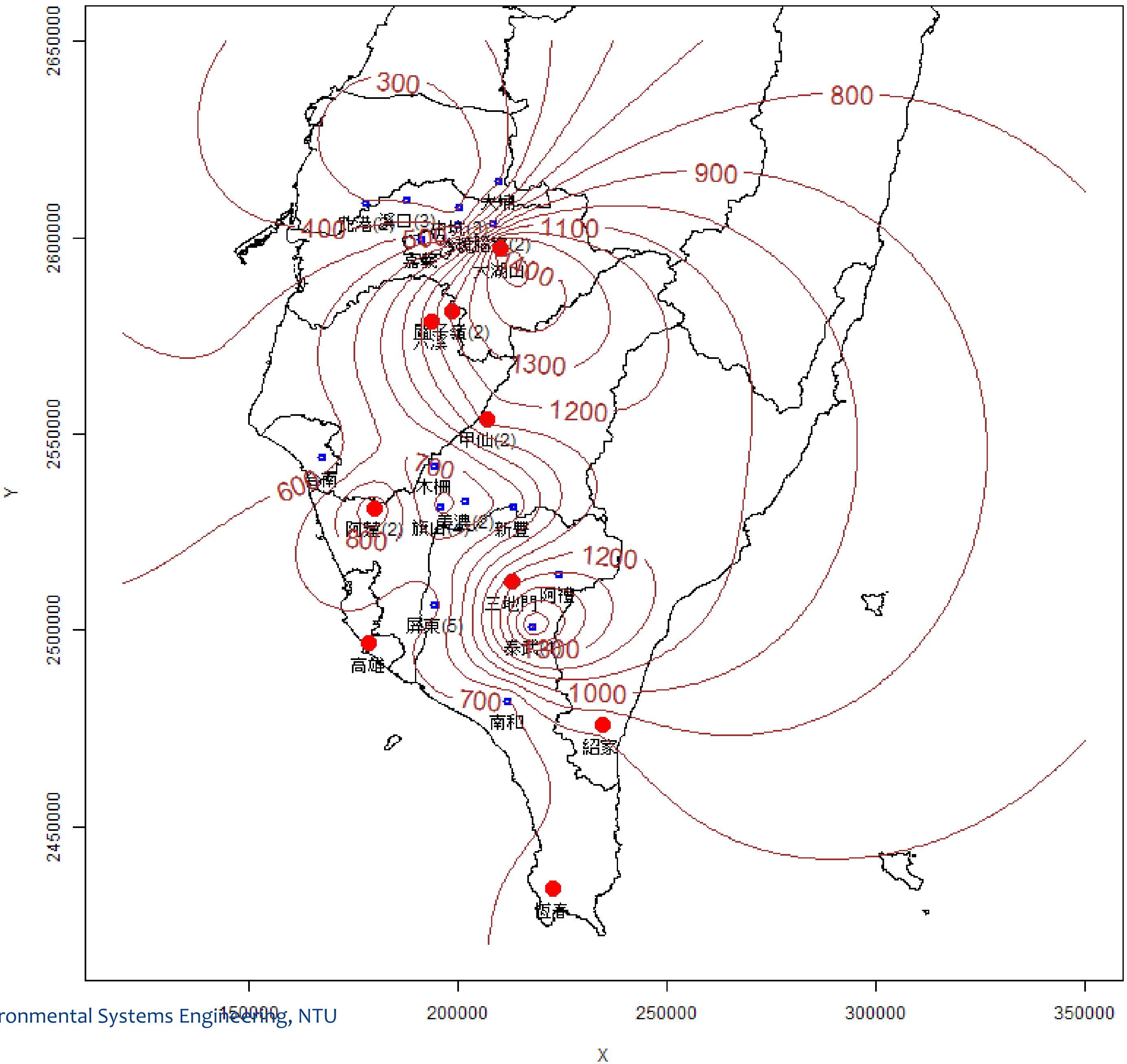


Type C

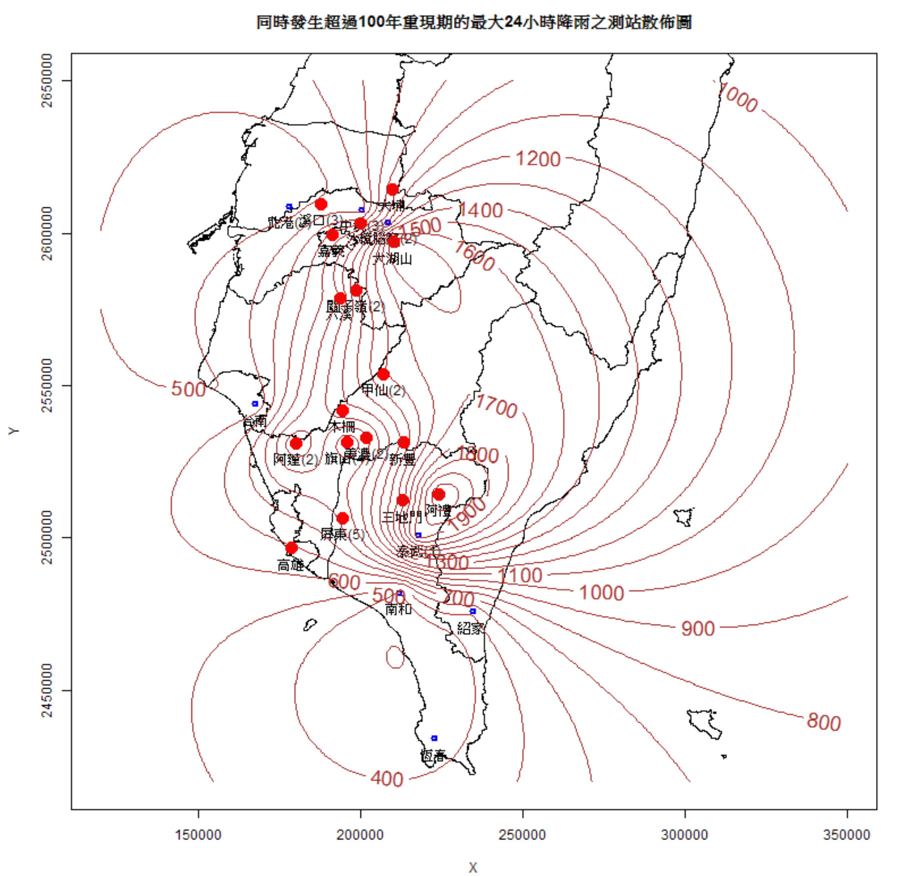


同時發生超過100年重現期的最大24小時降雨之測站散佈圖
500
900
300
1200
1300
1400
1500
1700
1800
1000
1100
800
700
600
200
大湖山
甲仙(2)
台南
阿蓮(2)
新豐
阿禮
屏東(5)
高雄
南和
2650000
2600000
2550000
2500000
2450000
Y
150000
200000
250000
300000
350000
X

同時發生超過100年重現期的最大24小時降雨之測站散佈圖

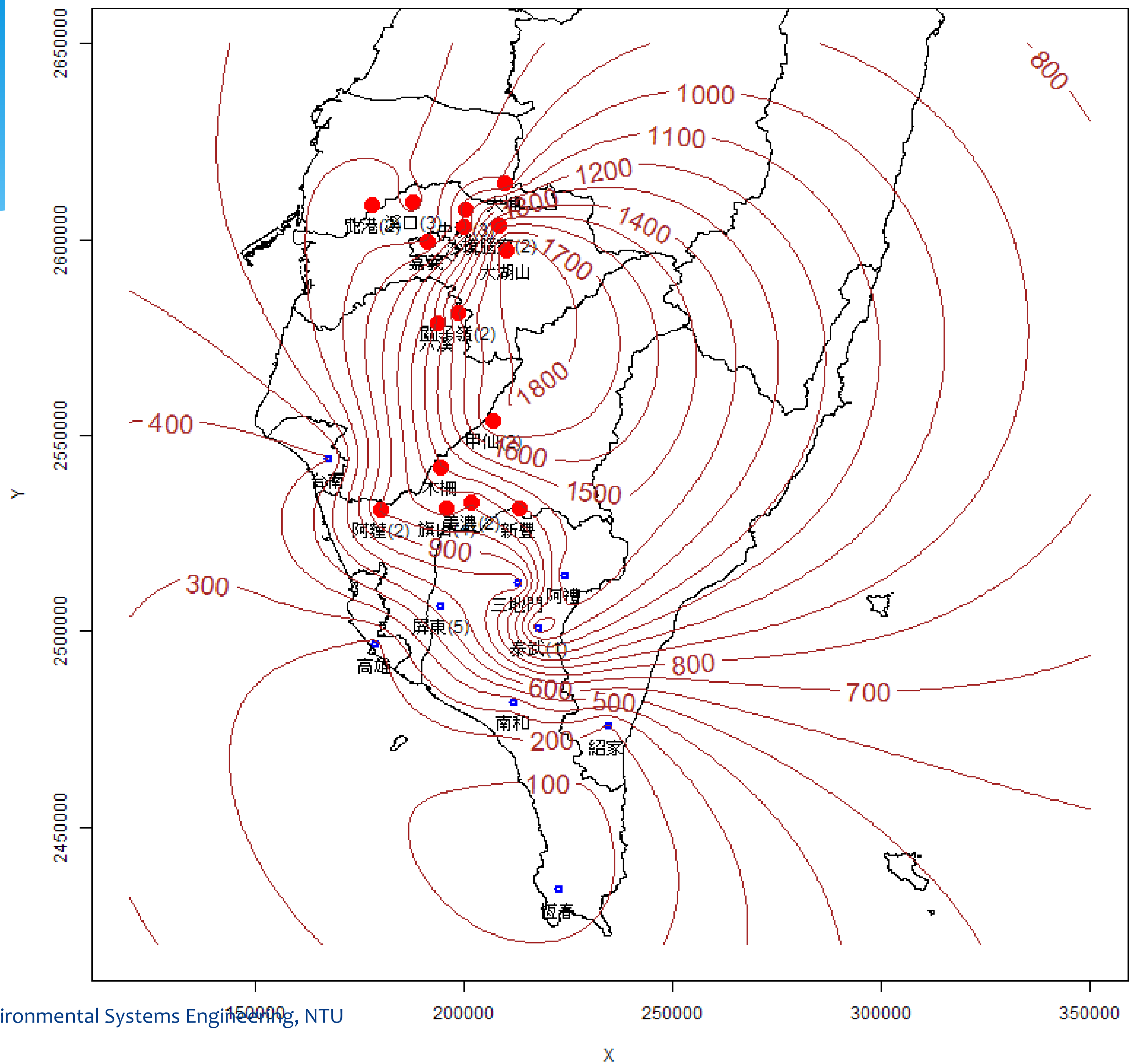


Type D



同時發生超過100年重現期的最大24小時降雨之測站散佈圖
2650000
2600000
2550000
2500000
2450000
Y
150000
200000
250000
300000
350000
X
1000
1200
1400
1500
1600
1700
1800
1900
1300
1100
1000
900
800
500
600
400
甲仙(2)
木柵
台南
阿蓮(2)
新豐
三地門
阿禮
屏東(5)
高雄
南和
紹家
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖



Summary

- We developed a stochastic approach for simulation of multi-site event-max rainfalls to cope with the problems of outliers and short record length in hydrological frequency analysis.
- By increasing the sample size and considering the spatial covariation of EMRs, the return periods of site-specific and multi-site rainfall extremes can be better estimated.

Suggestions

- **Establish an archive of storm events**
  - Seasonal and storm-type specific
  - Will be very helpful for meteorological and hydrological studies (climate change detection, freq. analysis, etc.)
- **Form a meteo-hydrology (hydro-meteology) working group**

Rationale of BVG simulation using frequency factor

- **From the view point of random number generation, the frequency factor can be considered as a random variable *K*, and K_T is a value of *K* with exceedence probability 1/*T*.**
- **Frequency factor of the Pearson type III distribution can be approximated by**

Standard normal deviate

$$K_T \approx z + (z^2 - 1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3 - 6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2 - 1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5 \quad [A]$$

- **General equation for hydrological frequency analysis**

$$X_T = \mu_X + K_T \sigma_X$$

Given μ_X, σ_X and γ_X, if we can generate a set of random numbers of K, say $k_1, k_2, \cdots, k_n$, then a random sample of X, say $x_1, x_2, \cdots, x_n$, can be obtained by $x_i = \mu_X + k_i \sigma_X$.

Note that each $k_i, i = 1,2,\cdots n$, corresponds to its own exceedence probability $1/T_i$.

- **The gamma distribution is a special case of the Pearson type III distribution with a zero location parameter. Therefore, it seems plausible to generate random samples of a bivariate gamma distribution based on two jointly distributed frequency factors.**

$$K_T \approx z + \left(z^2 - 1\right)\frac{\gamma_X}{6} + \frac{1}{3}\left(z^3 - 6z\right)\left(\frac{\gamma_X}{6}\right)^2 - \left(z^2 - 1\right)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5 \qquad [A]$$

Gamma density

$$f_X(x\,;\alpha,\beta)=\frac{1}{\alpha\Gamma(\beta)}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-x/\alpha},\quad 0\le x<+\infty$$

$$\alpha=\frac{\sigma}{\sqrt{\beta}}>0 \qquad \beta=\left(\frac{2}{\gamma}\right)^2>0 \qquad \mu=\alpha\beta=\sigma\sqrt{\beta}>0$$

μ, σ, and γ are the mean, standard deviation, and skewness coefficient of X (or Y), respectively, and α and β are respectively the scale and shape parameters of the gamma density.

$$\sigma=\frac{\mu\gamma}{2}$$

$$K_T \approx z + (z^2-1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3-6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2-1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

- **Assume two gamma random variables *X* and *Y* are jointly distributed.**
- **The two random variables are respectively associated with their frequency factors K_X and K_Y.**
- **Equation (A) indicates that the frequency factor K_X of a random variable *X* with gamma density is approximated by a function of the standard normal deviate and the coefficient of skewness of the gamma density.**

$$K_T \approx z + (z^2 - 1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3 - 6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2 - 1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

Simulation of the frequency factor K_X can be achieved by generating a random sample of the standard normal deviate U, say $u_1, u_2, \cdots, u_n$, and then utilizing Eq. (A) to obtain $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ from $u_1, u_2, \cdots, u_n$.

However, for a bivariate gamma density $f_{XY}(x, y)$, the two frequency factors K_X and K_Y are correlated through two correlated standard normal deviates U and V, with a correlation coefficient ρ_{UV}.

- **Thus, random number generation of the second frequency factor K_Y must take into consideration the correlation between K_X and K_Y which stems from the correlation between U and V.**

Conditional normal density

- **Given a random number of *U*, say *u*, the conditional density of *V* is expressed by the following conditional normal density**

$$\phi_{V|U}(v|U=u)=\frac{1}{\sqrt{2\pi(1-\rho_{UV}^2)}}\cdot\exp\left\{-\frac{1}{2}\left[\frac{v-\rho_{UV}u}{\sqrt{1-\rho_{UV}^2}}\right]^2\right\}$$

with mean $\rho_{UV}u$ **and variance** $1-\rho_{UV}^2$**.**

Thus, based on a random sample $u_1, u_2, \cdots, u_n$ of U, a random sample of V, say $v_1, v_2, \cdots, v_n$, can be generated by a normal random number generator with means $\rho_{UV} u_i \ (i = 1, 2, \cdots, n)$ and variance $1 - \rho_{UV}^2$.

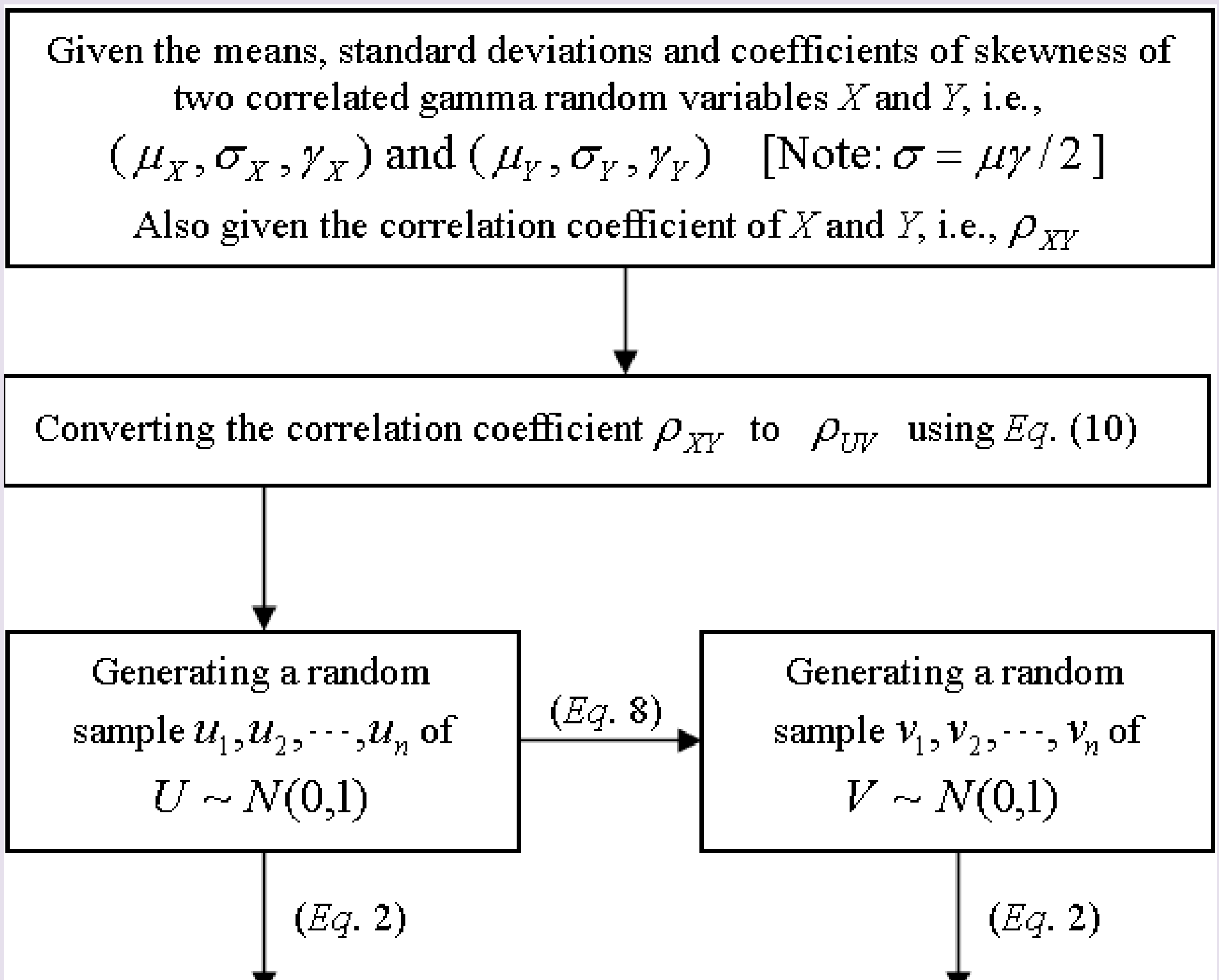
$$\phi_{V|U}(v \mid U = u) = \frac{1}{\sqrt{2\pi(1-\rho_{UV}^2)}} \cdot \exp\left\{-\frac{1}{2}\left[\frac{v - \rho_{UV} u}{\sqrt{1-\rho_{UV}^2}}\right]^2\right\}$$

$$X = \mu_X + K\sigma_X$$

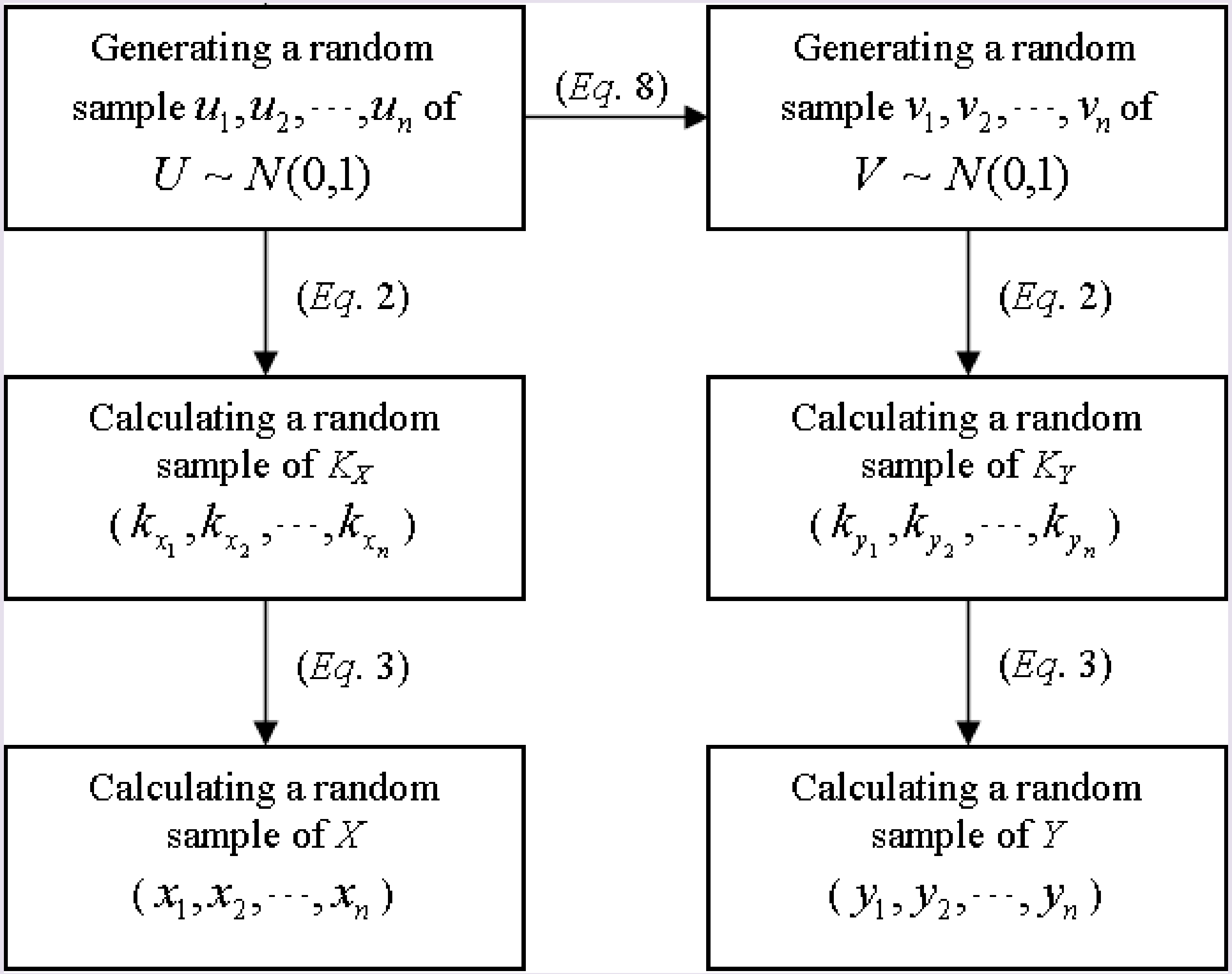
From the two sets of random samples $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, Eq. (A) can be used to obtain random samples of the two frequency factors K_X and K_Y, i.e., $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ and $k_{y_1}, k_{y_2}, \cdots, k_{y_n}$.

Given the expected values (μ_X and μ_Y) and standard deviations (σ_X and σ_Y) of random variables X and Y, random samples of the bivariate gamma distribution can be obtained by transferring $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ and $k_{y_1}, k_{y_2}, \cdots, k_{y_n}$ to $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$.

Flowchart of BVG simulation (1/2)



Flowchart of BVG simulation (2/2)



In practice, stochastic simulation of a bivariate gamma distribution requires the generated random samples to have pre-specified mean, standard deviation, coefficient of skewness (($\mu_X, \sigma_X, \gamma_X$) and ($\mu_Y, \sigma_Y, \gamma_Y$)), and correlation coefficient ρ_{XY}. In order for the generated samples to meet such requirements, the correlation coefficient ρ_{UV} must be determined from the pre-specified γ_X, γ_Y, and ρ_{XY} through the following equation:

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$

$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3 \qquad [B]$$

$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \qquad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \qquad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \qquad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \qquad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

Derivation of the $\rho_{XY} \sim \rho_{UV}$ relationship

Suppose that random variables X and Y form a bivariate gamma distribution. Given the means (μ_X and μ_Y) and standard deviations (σ_X and σ_Y), X and Y can be respectively expressed in terms of their corresponding frequency factors K_X and K_Y , i.e.,

$$X = \mu_X + K_X \sigma_X \quad \text{and} \quad Y = \mu_Y + K_Y \sigma_Y \,.$$

Note that, with given means μ_X and μ_Y and standard deviations σ_X and σ_Y , the coefficients of skewness γ_X and γ_Y can be readily determined.

From the above equations, it can be easily shown that frequency factors K_X and K_Y are distributed with zero mean and unit standard deviation, and correlation coefficient of X and Y is equivalent to correlation coefficient of K_X and K_Y, i.e.,

$$E[K_X] = E[K_Y] = 0,$$

$$Var[K_X] = Var[K_Y] = 1,$$

and

$$\rho_{XY} = \rho_{K_X K_Y}$$

- **Frequency factors K_X and K_Y can be respectively approximated by**

$$K_X \approx U + \left(U^2 - 1\right)\frac{\gamma_X}{6} + \frac{1}{3}\left(U^3 - 6U\right)\left(\frac{\gamma_X}{6}\right)^2 - \left(U^2 - 1\right)\left(\frac{\gamma_X}{6}\right)^3 + U\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$K_Y \approx V + \left(V^2 - 1\right)\frac{\gamma_Y}{6} + \frac{1}{3}\left(V^3 - 6V\right)\left(\frac{\gamma_Y}{6}\right)^2 - \left(V^2 - 1\right)\left(\frac{\gamma_Y}{6}\right)^3 + V\left(\frac{\gamma_Y}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

where U and V both are random variables with standard normal density and are correlated with correlation coefficient ρ_{UV} .

- **Correlation coefficient of *K_X* and *K_Y* can be derived as follows:**

$$\rho_{K_X K_Y} = Cov(K_X, K_Y) = E\left[K_X K_Y\right]$$

$$\approx E\left\{\begin{array}{l}\left[U + \left(U^2 - 1\right)\frac{\gamma_X}{6} + \frac{1}{3}\left(U^3 - 6U\right)\left(\frac{\gamma_X}{6}\right)^2 - \left(U^2 - 1\right)\left(\frac{\gamma_X}{6}\right)^3 + U\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5\right] \cdot \\ \left[V + \left(V^2 - 1\right)\frac{\gamma_Y}{6} + \frac{1}{3}\left(V^3 - 6V\right)\left(\frac{\gamma_Y}{6}\right)^2 - \left(V^2 - 1\right)\left(\frac{\gamma_Y}{6}\right)^3 + V\left(\frac{\gamma_Y}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5\right]\end{array}\right\}$$

$$K_X \approx U\left[1+\left(\frac{\gamma_X}{6}\right)^4\right]+\left(U^2-1\right)\left[\frac{\gamma_X}{6}-\left(\frac{\gamma_X}{6}\right)^3\right]+\frac{1}{3}\left(U^3-6U\right)\left(\frac{\gamma_X}{6}\right)^2-\frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$= A_X U + B_X\left(U^2-1\right)+C_X\left(U^3-6U\right)+D_X$$

$$K_Y \approx V\left[1+\left(\frac{\gamma_Y}{6}\right)^4\right]+\left(V^2-1\right)\left[\frac{\gamma_Y}{6}-\left(\frac{\gamma_Y}{6}\right)^3\right]+\frac{1}{3}\left(V^3-6V\right)\left(\frac{\gamma_Y}{6}\right)^2-\frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

$$= A_Y V + B_Y\left(V^2-1\right)+C_Y\left(V^3-6V\right)+D_Y$$

$$A_X = 1+\left(\frac{\gamma_X}{6}\right)^4,\ B_X = \frac{\gamma_X}{6}-\left(\frac{\gamma_X}{6}\right)^3,\ C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2,\ D_X = -\frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$A_Y = 1+\left(\frac{\gamma_Y}{6}\right)^4,\ B_Y = \frac{\gamma_Y}{6}-\left(\frac{\gamma_Y}{6}\right)^3,\ C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2,\ D_Y = -\frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

$$
\begin{aligned}
&E[K_X K_Y] \\
&\approx E\begin{bmatrix}
A_X A_Y UV + A_X B_Y U(V^2-1) + A_X C_Y U(V^3-6V) \\
+ A_X D_Y U + B_X A_Y V(U^2-1) + B_X B_Y (U^2-1)(V^2-1) \\
+ B_X C_Y (U^2-1)(V^3-6V) + B_X D_Y (U^2-1) \\
+ C_X A_Y (U^3-6U)V + C_X B_Y (U^3-6U)(V^2-1) \\
+ C_X C_Y (U^3-6U)(V^3-6V) + C_X D_Y (U^3-6U) \\
+ D_X A_Y V + D_X B_Y (V^2-1) + D_X C_Y (V^3-6V) + D_X D_Y
\end{bmatrix}
\end{aligned}
$$

$$E[K_X K_Y]$$

$$\approx E\begin{bmatrix} A_X A_Y UV + A_X C_Y U(V^3 - 6V) + B_X B_Y (U^2 - 1)(V^2 - 1) \\ + C_X A_Y (U^3 - 6U)V + C_X C_Y (U^3 - 6U)(V^3 - 6V) + D_X D_Y \end{bmatrix}$$

$$E[K_X] \approx E[A_X U + B_X (U^2 - 1) + C_X (U^3 - 6U) + D_X] = D_X$$

Since K_X and K_Y are distributed with zero means, it follows that

$$D_X = D_Y \approx 0$$

$$\rho_{K_X K_Y} = E\left[K_X K_Y\right]$$

$$\approx E\left[\begin{array}{l} A_X A_Y UV + A_X C_Y U\left(V^3 - 6V\right) + B_X B_Y\left(U^2 - 1\right)\left(V^2 - 1\right) \\ + C_X A_Y\left(U^3 - 6U\right)V + C_X C_Y\left(U^3 - 6U\right)\left(V^3 - 6V\right) \end{array}\right]$$

$$= A_X A_Y \rho_{UV} + A_X C_Y\left[E\left(UV^3\right) - 6\rho_{UV}\right] + B_X B_Y\left[E\left(U^2 V^2\right) - 1\right]$$

$$+ C_X A_Y\left[E\left(U^3 V\right) - 6\rho_{UV}\right]$$

$$+ C_X C_Y\left[E\left(U^3 V^3\right) - 6E\left(UV^3\right) - 6E\left(U^3 V\right) + 36\rho_{UV}\right]$$

- **It can also be shown that**

$$E\left(U^2V^2\right) = 2\rho_{UV}^2 + 1 \qquad E\left(U^3V^3\right) = 6\rho_{UV}^3 + 9\rho_{UV}$$

$$E\left(U^3V\right) = E\left(UV^3\right) = 3\rho_{UV}$$

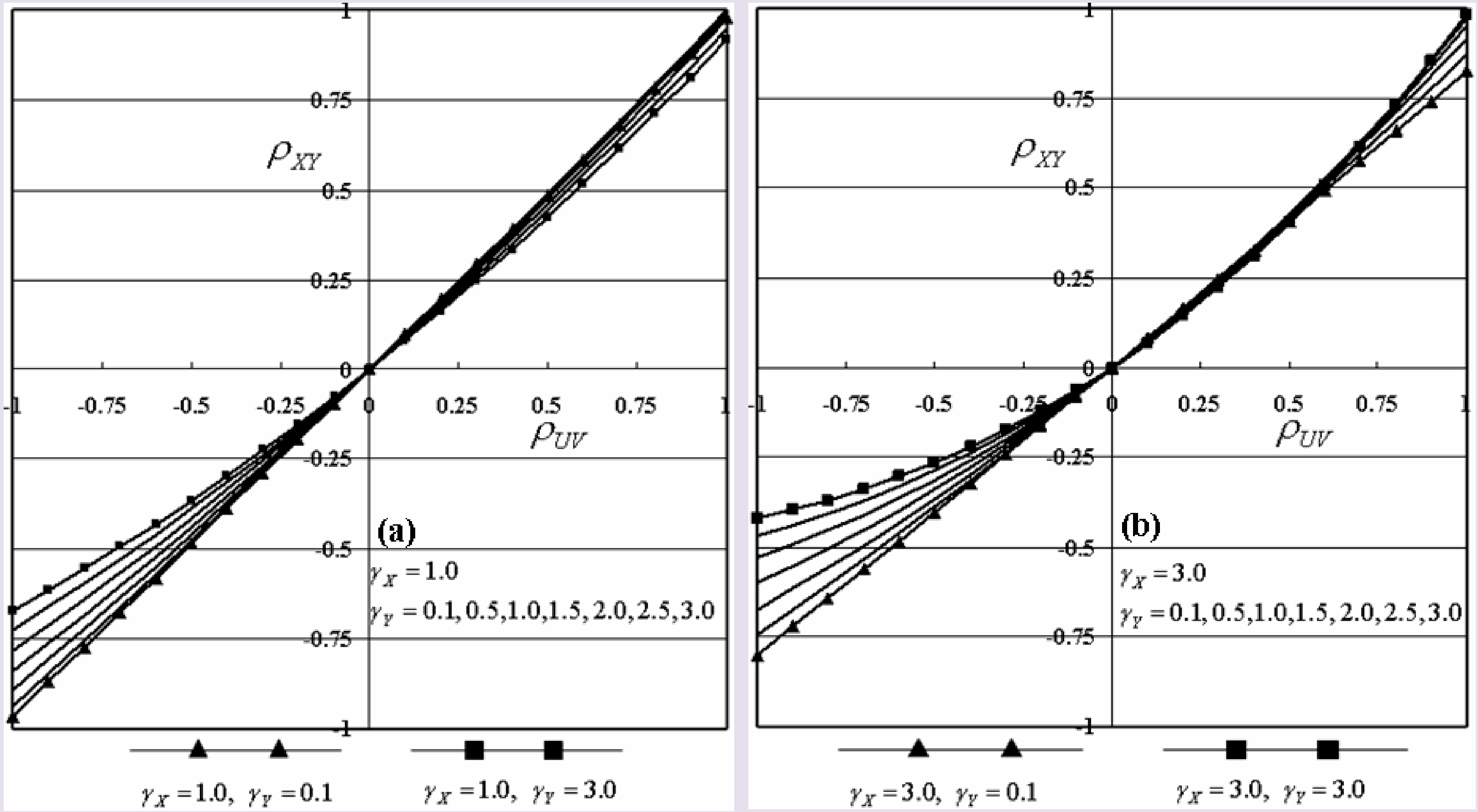
Thus,

$$\rho_{XY} = \rho_{K_X K_Y} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$

$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$

Equation (B) indicates that ρ_{XY} can be expressed as a third order polynomial of ρ_{UV} . It is therefore of practical concern whether there exits a unique ρ_{UV} for a given set of (γ_X , γ_Y , ρ_{XY}). Or equivalently, given a set of (γ_X , γ_Y , ρ_{XY}), does Eq. (B) return a single-value of ρ_{UV} ?

$\rho_{XY} \sim \rho_{UV}$ Single-Value Relationship



We have also proved that Eq. (B) is indeed a single-value function.

Proof of Eq. (B) as a single-value function

Let $f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV}$. From Eq. (B) we have

$$f(\rho_{UV}) = (A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y)$$
$$+ 4B_X B_Y \rho_{UV} + 18C_X C_Y \rho_{UV}^2$$

$$A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y = (A_X - 3C_X)(A_Y - 3C_Y)$$

$$A_X - 3C_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 - \left(\frac{\gamma_X}{6}\right)^2 = \left[\left(\frac{\gamma_X}{6}\right)^2 - 1\right]^2 + \left(\frac{\gamma_X}{6}\right)^2 > 0$$

$$A_Y - 3C_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 - \left(\frac{\gamma_Y}{6}\right)^2 = \left[\left(\frac{\gamma_Y}{6}\right)^2 - 1\right]^2 + \left(\frac{\gamma_Y}{6}\right)^2 > 0 .$$

- **Therefore,**

$$A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y$$

$$= \left\{ \left[\left(\frac{\gamma_X}{6} \right)^2 - 1 \right]^2 + \left(\frac{\gamma_X}{6} \right)^2 \right\} \left\{ \left[\left(\frac{\gamma_Y}{6} \right)^2 - 1 \right]^2 + \left(\frac{\gamma_Y}{6} \right)^2 \right\}$$

Let $g(\rho_{UV}) = 4B_X B_Y \rho_{UV} + 18C_X C_Y \rho_{UV}^2$

$$= \frac{\gamma_X \gamma_Y}{9}\left[\left(\frac{\gamma_X}{6}\right)^2 - 1\right]\left[\left(\frac{\gamma_Y}{6}\right)^2 - 1\right]\rho_{UV} + \frac{18}{9}\left(\frac{\gamma_X}{6}\right)^2\left(\frac{\gamma_Y}{6}\right)^2 \rho_{UV}^2$$

Also, let $G_X = \frac{\gamma_X}{6}$, $G_Y = \frac{\gamma_Y}{6}$. We then have

$$g(\rho_{UV}) = 4G_X G_Y (G_X^2 - 1)(G_Y^2 - 1)\rho_{UV} + 2G_X^2 G_Y^2 \rho_{UV}^2$$

$$f(\rho_{UV}) = \left[(G_X^2 - 1)^2 + G_X^2\right]\left[(G_Y^2 - 1)^2 + G_Y^2\right]$$
$$+ 4G_X G_Y (G_X^2 - 1)(G_Y^2 - 1)\rho_{UV} + 2G_X^2 G_Y^2 \rho_{UV}^2$$

$$\frac{\partial f}{\partial \rho_{UV}} = 4G_X G_Y (G_X^2 - 1)(G_Y^2 - 1) + 4G_X^2 G_Y^2 \rho_{UV}$$

Let $\frac{\partial f}{\partial \rho_{UV}} = 0$, it yields an extreme value of f at

$$\rho_{UV}^* = -\frac{(G_X^2 - 1)(G_Y^2 - 1)}{G_X G_Y}$$

$$f(\rho_{UV}^{*}) = \left[(G_X^2-1)^2+G_X^2\right]\left[(G_Y^2-1)^2+G_Y^2\right]$$
$$-4(G_X^2-1)^2(G_Y^2-1)^2+2(G_X^2-1)^2(G_Y^2-1)^2$$

$$=\left[(G_X^2-1)^2+G_X^2\right]\left[(G_Y^2-1)^2+G_Y^2\right]-2(G_X^2-1)^2(G_Y^2-1)^2$$

$$=G_X^2(G_Y^2-1)^2+(G_X^2-1)^2G_Y^2+G_X^2G_Y^2-(G_X^2-1)^2(G_Y^2-1)^2$$

Since $-1 \le \rho_{UV} \le 1$ (or equivalently, $\rho_{UV}^2 \le 1$), it yields

$$(G_X^2-1)^2(G_Y^2-1)^2 \le G_X^2G_Y^2$$

Thus,

$$f(\rho_{UV}^{*}) \ge G_X^2(G_Y^2-1)^2+(G_X^2-1)^2G_Y^2 > 0$$

We now check the second derivative of $f(\rho_{UV})$, i.e.,

$$\frac{\partial^2 f}{\partial(\rho_{UV})^2} = 4G_X^2 G_Y^2 > 0$$

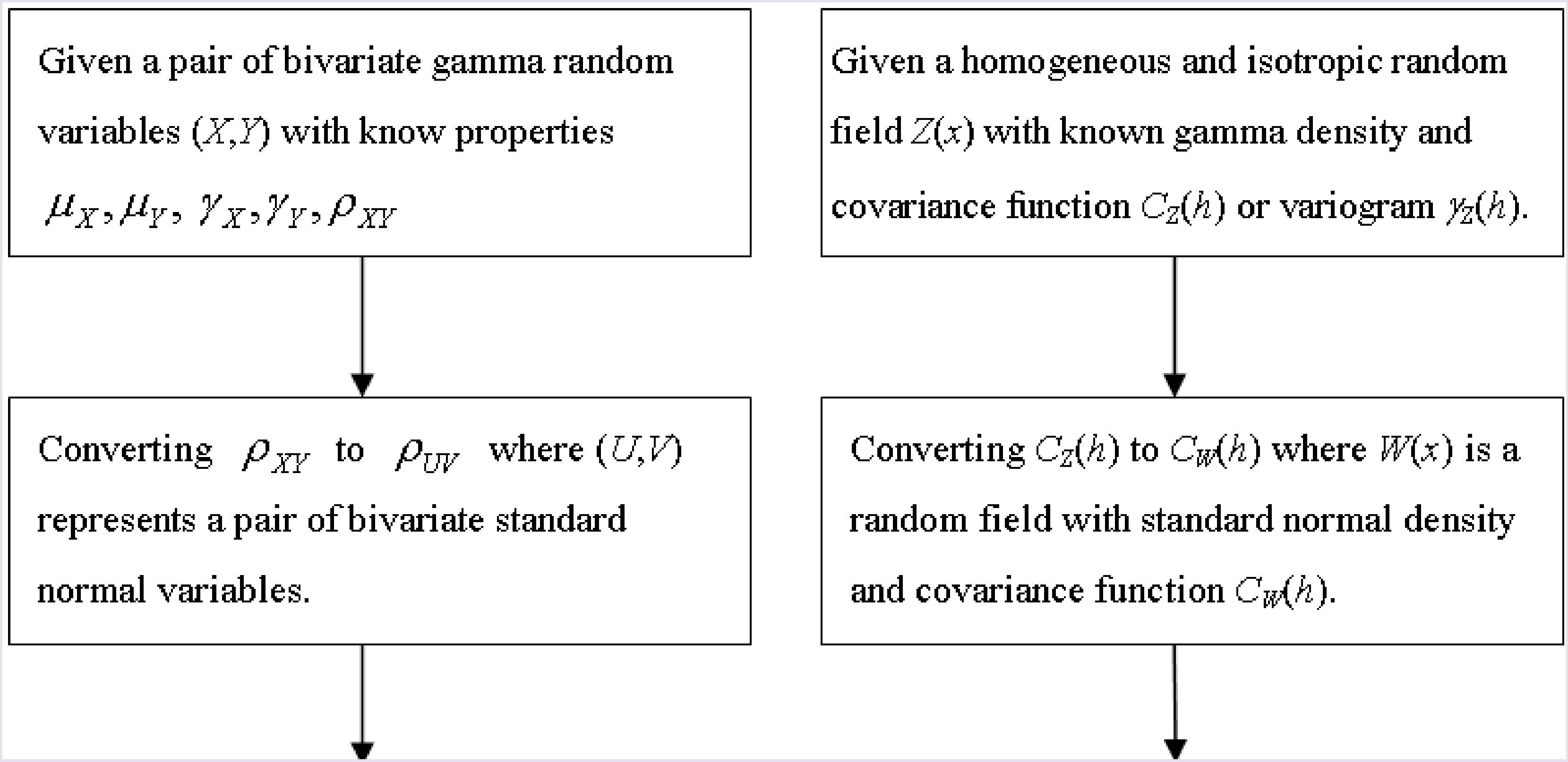
Therefore, $f(\rho_{UV}^*) > 0$ is the minimum of the function $f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV}$. It follows that

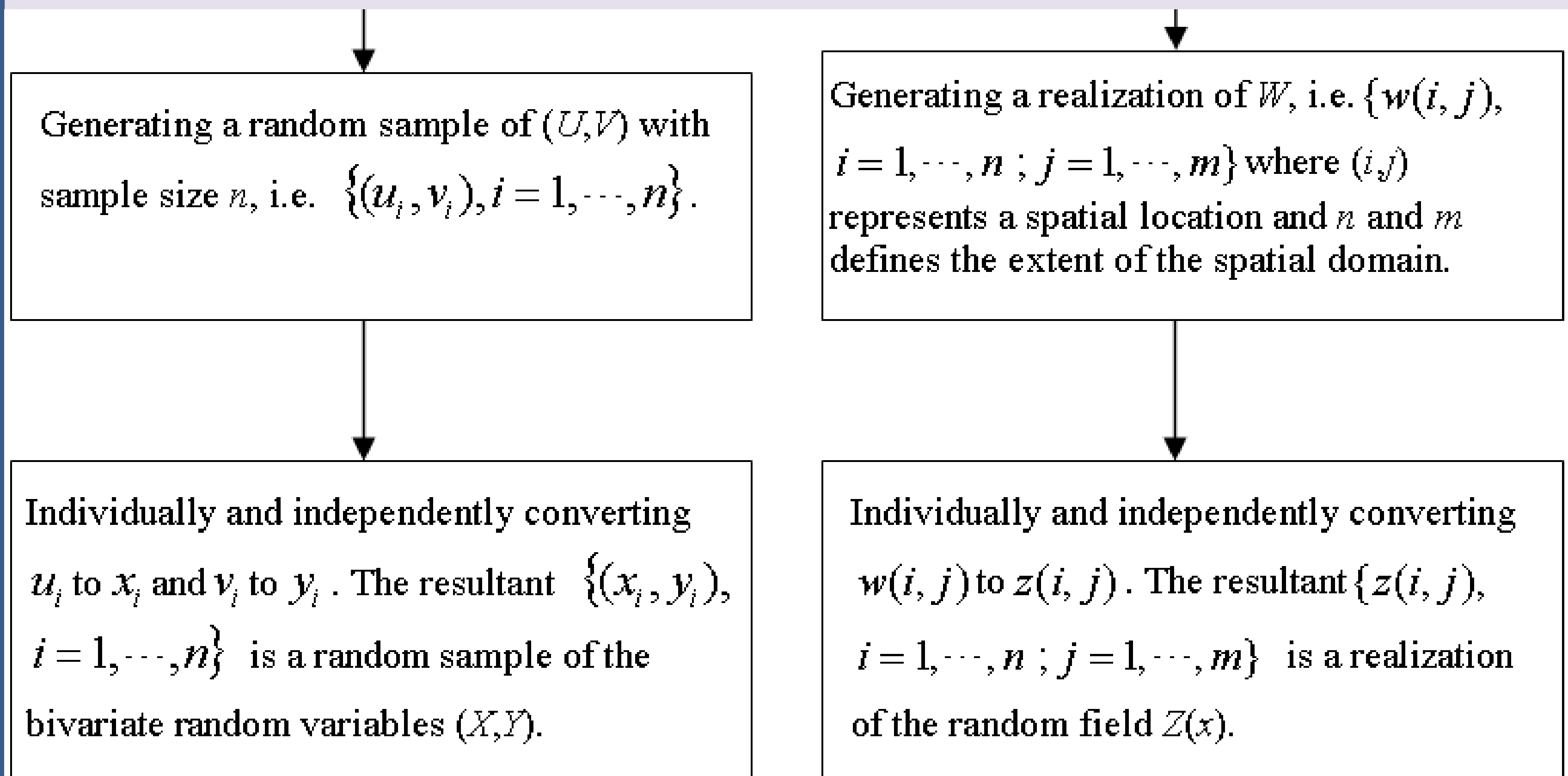
$$f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV} > 0$$

for all possible values of ρ_{UV}.

- **The above equation indicates ρ_{XY} increases with increasing ρ_{UV} , and thus Eq. (B) is a single-value function.**

Conceptual description of a gamma random field simulation approach




Generating a random sample of (U,V) with sample size n, i.e. $\{(u_i, v_i), i = 1, \cdots, n\}$.
Generating a realization of W, i.e. $\{w(i, j), i = 1, \cdots, n\ ;\ j = 1, \cdots, m\}$ where (i,j) represents a spatial location and n and m defines the extent of the spatial domain.
Individually and independently converting u_i to x_i and v_i to y_i . The resultant $\{(x_i, y_i), i = 1, \cdots, n\}$ is a random sample of the bivariate random variables (X,Y).
Individually and independently converting $w(i, j)$ to $z(i, j)$. The resultant $\{z(i, j), i = 1, \cdots, n\ ;\ j = 1, \cdots, m\}$ is a realization of the random field $Z(x)$.