Use Riemann Solver for Atmospheric Modeling

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105年氣象局天氣分析研討會

Introduction

- Modeling atmosphere requires to deal with
  - CFL condition: numerical instability
    - Stable advection scheme or semi-Lagrangian advection
  - High frequency modes (terms)
    - Time-split on different terms or semi-implicit time integration
- Solve both independently
  - Eulerian model, time-split on terms or semi-implicit
  - Semi-Lagrangian advection in grid-point model
- Solve both together
  - Semi-implicit semi-Lagrangian scheme in spectral model
  - Semi-Lagrangian advection along high frequent wave

Contents

- Introducing Riemann Solver by Riemann invariant characteristic equation (RICE)
  - Simple 1D for easy illustration
- Possible 2D
  - In shallow water equation
  - In nonhydrostatic system
- How about 3D?
  - With splitting, 2D then 1D in vertical
- Discussion
  - advantage
  - Future work

On dimensional shallow water equation (gravity waves)

$$\frac{\partial h}{\partial t}+\frac{\partial(uh)}{\partial x}=0 \qquad \& \qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=0$$

can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \end{pmatrix}+\begin{pmatrix} u & h \\ g & u \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} h \\ u \end{pmatrix}=0$$

And we can have a matrix L and it inverse matrix L^{-1} for the above matrix be diagonal matrix with eigenvector as

$$L^{-1}\begin{pmatrix} u & h \\ g & u \end{pmatrix}L=\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \qquad \& \qquad LL^{-1}=1$$

After we solve L, put it into above equation we have

$$L^{-1}\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \end{pmatrix} + L^{-1}\begin{pmatrix} u & h \\ g & u \end{pmatrix} L L^{-1}\frac{\partial}{\partial x}\begin{pmatrix} h \\ u \end{pmatrix} = 0$$

then the shallow water equation can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0$$

or

$$\frac{\partial R_1}{\partial t} + C_1\frac{\partial R_1}{\partial x} = 0$$

$$\frac{\partial R_2}{\partial t} + C_2\frac{\partial R_2}{\partial x} = 0$$

where

$$R_1 = \sqrt{gh} + u/2$$

$$R_2 = \sqrt{gh} - u/2$$

$$C_1 = u + \sqrt{gh}$$

$$C_2 = u - \sqrt{gh}$$

So the procedure to solve the 1D SWE by

1) Obtain R and C from u and h at any model grid as

$$R_1(t) = \sqrt{gh(t)} + u(t)/2 \qquad C_1(t) = u(t) + \sqrt{gh(t)}$$

&

$$R_2(t) = \sqrt{gh(t)} - u(t)/2 \qquad C_2(t) = u(t) - \sqrt{gh(t)}$$

2) Use advection eq to solve next time step of R by

$$\frac{\partial R_i}{\partial t} + C_i \frac{\partial R_i}{\partial x} = 0$$

to get

$$R_1(t+\Delta t) \quad and \quad R_2(t+\Delta t)$$

3) Obtain next time step u and h and C by

$$u(t+\Delta t) = R_1(t+\Delta t) - R_2(t+\Delta t)$$

$$h(t+\Delta t) = \frac{1}{g}\left(\frac{R_1(t+\Delta t) + R_2(t+\Delta t)}{2}\right)^2$$

&

$$C_1(t+\Delta t) = u(t+\Delta t) + \sqrt{gh(t+\Delta t)}$$

$$C_2(t+\Delta t) = u(t+\Delta t) - \sqrt{gh(t+\Delta t)}$$

4) Back to 2) for the next time

Select 1D case

- Follow Toda et al 2009 (JCP) weak nonlinear case
- Initial condition as

$$h(x,t=0)=1.0+0.01\exp\left\{-\left(\frac{x-x_m/2}{5}\right)^2\right\}$$

From Toda et al 2009

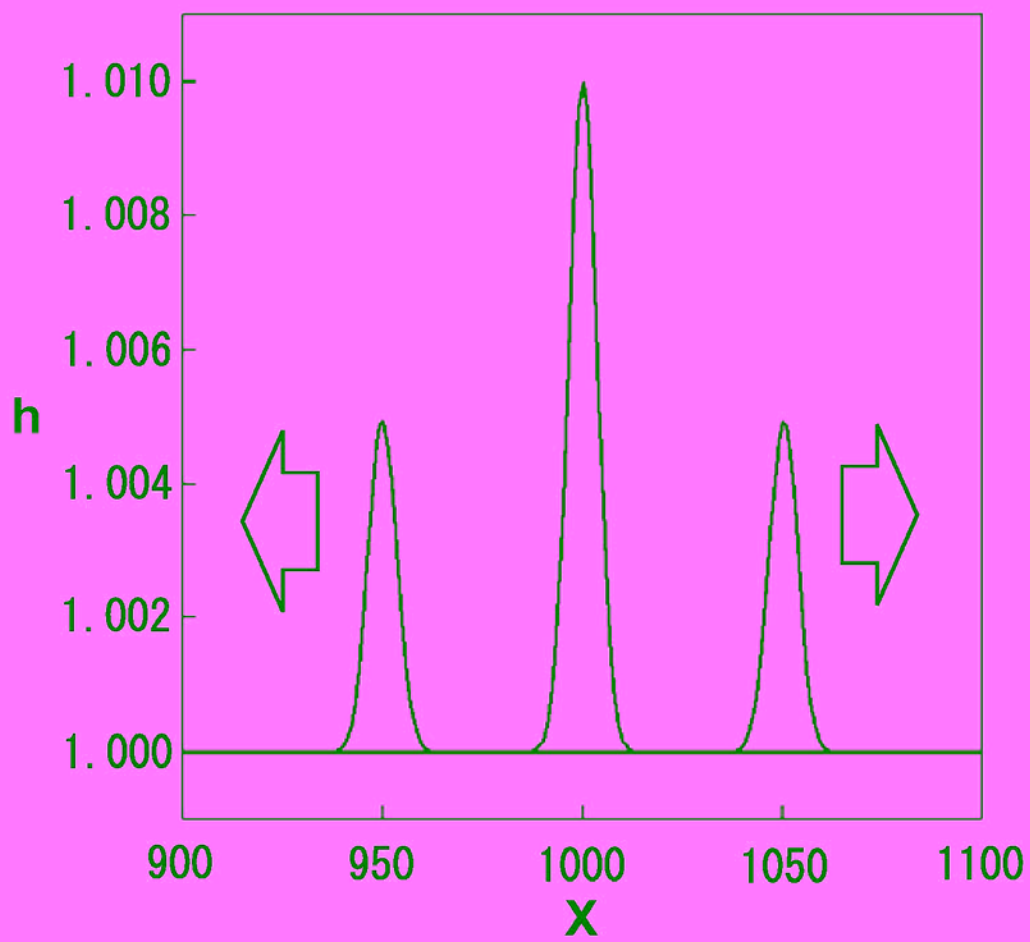


Fig. 6. Initial condition and solution behavior.

From Toda et al 2009

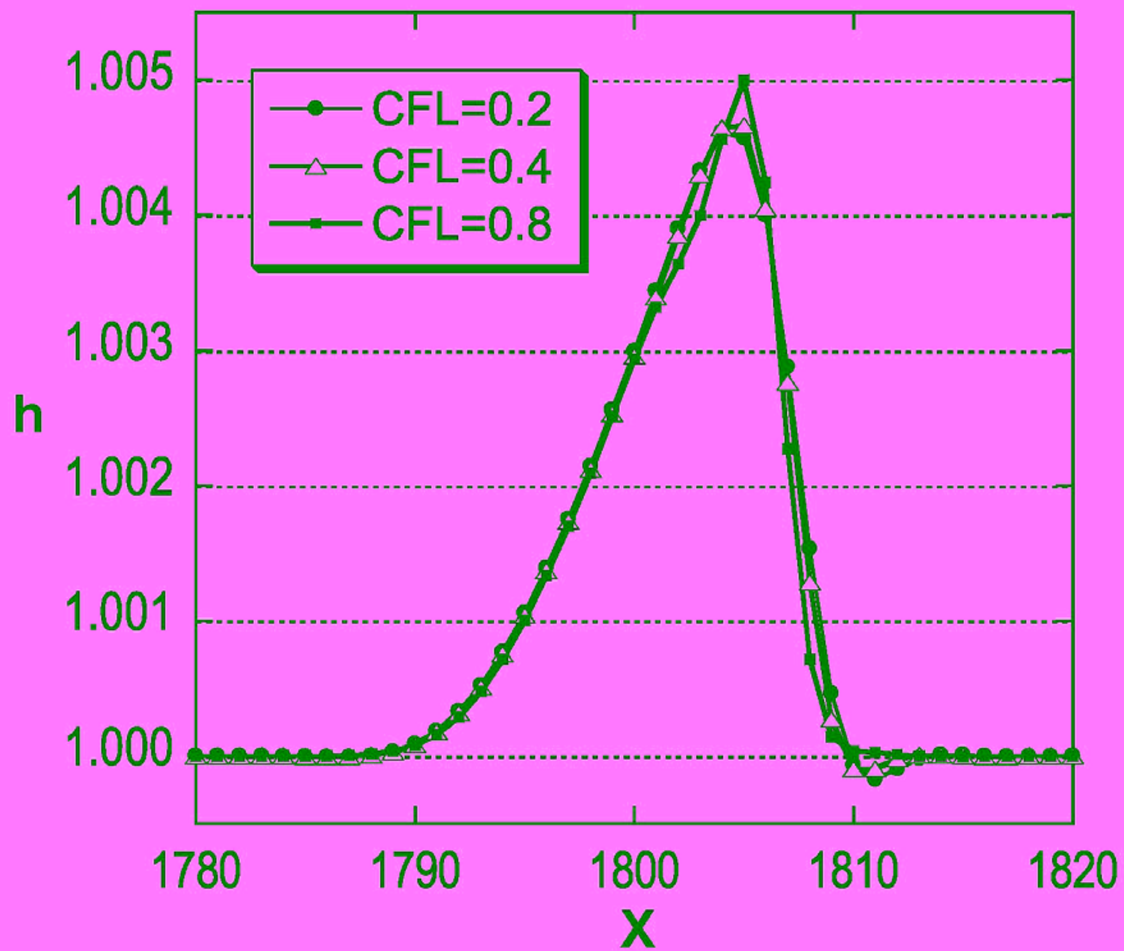


Fig. 9. Difference by CFL.

From Toda et al 2009

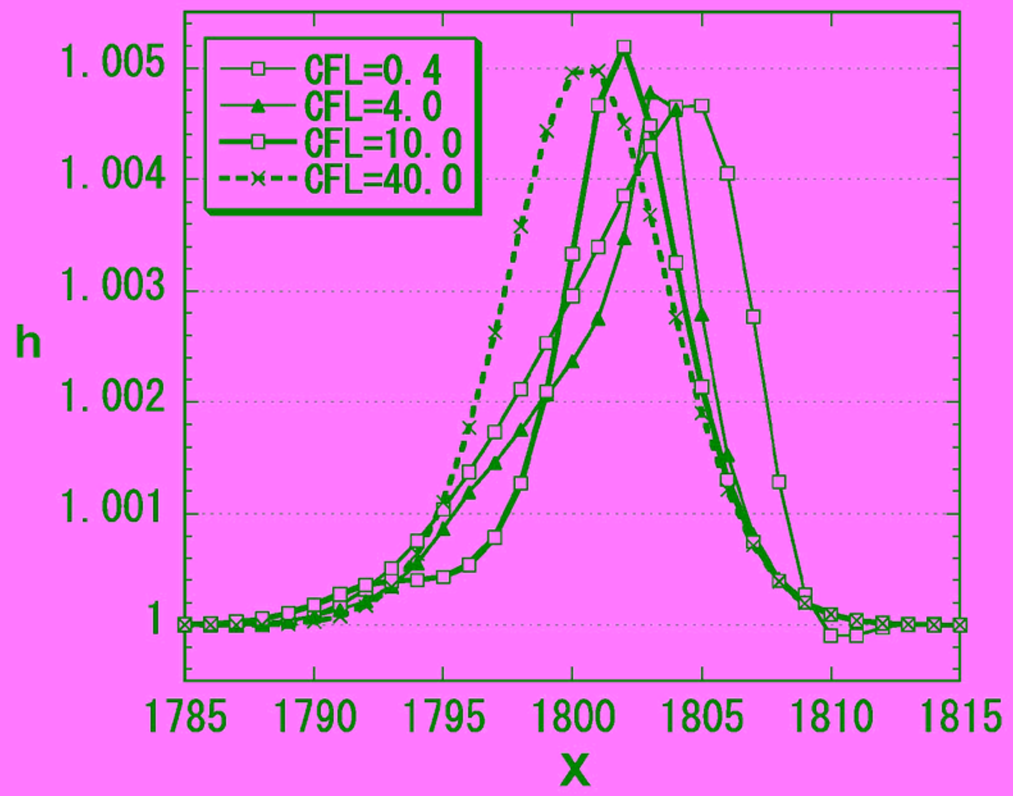
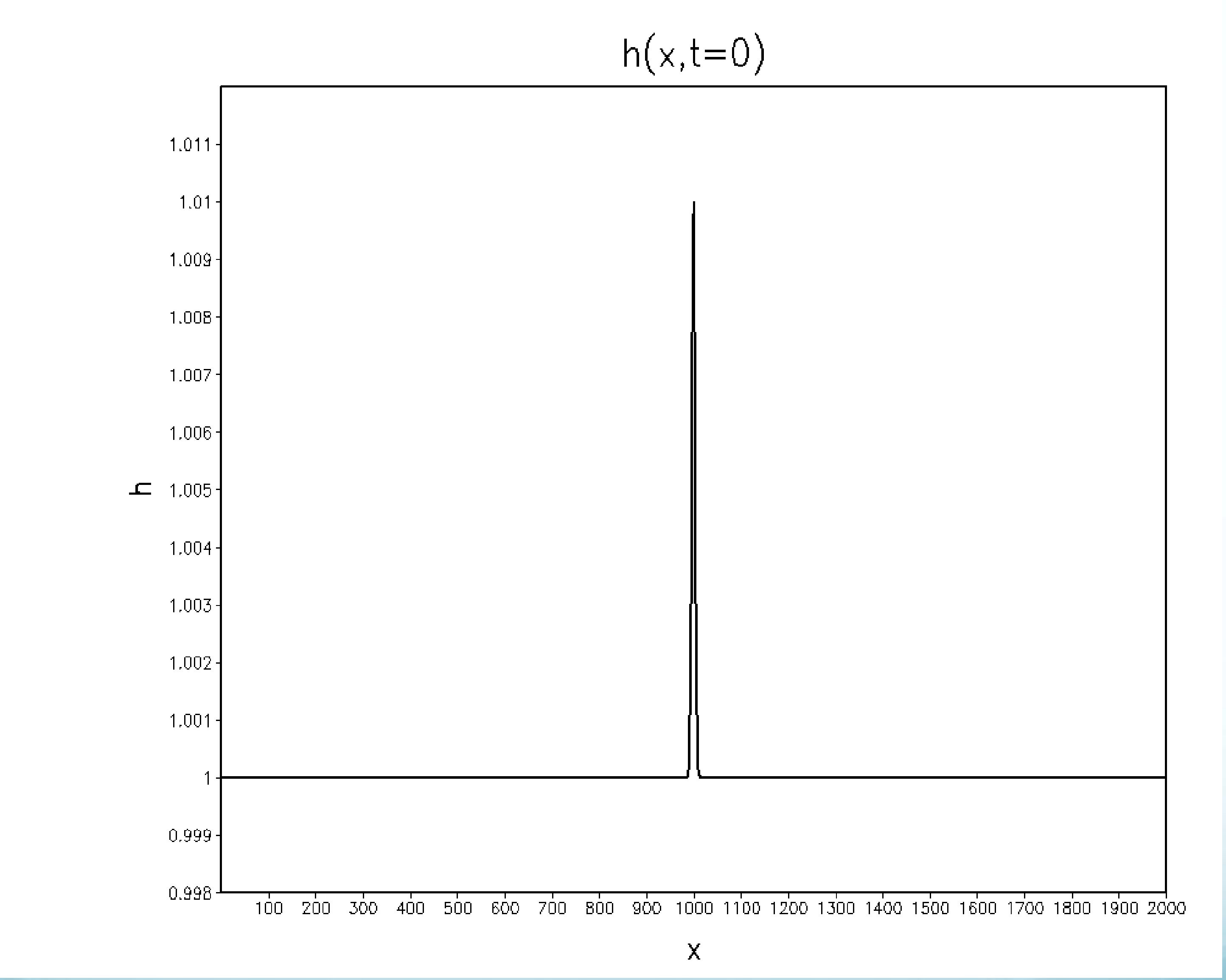
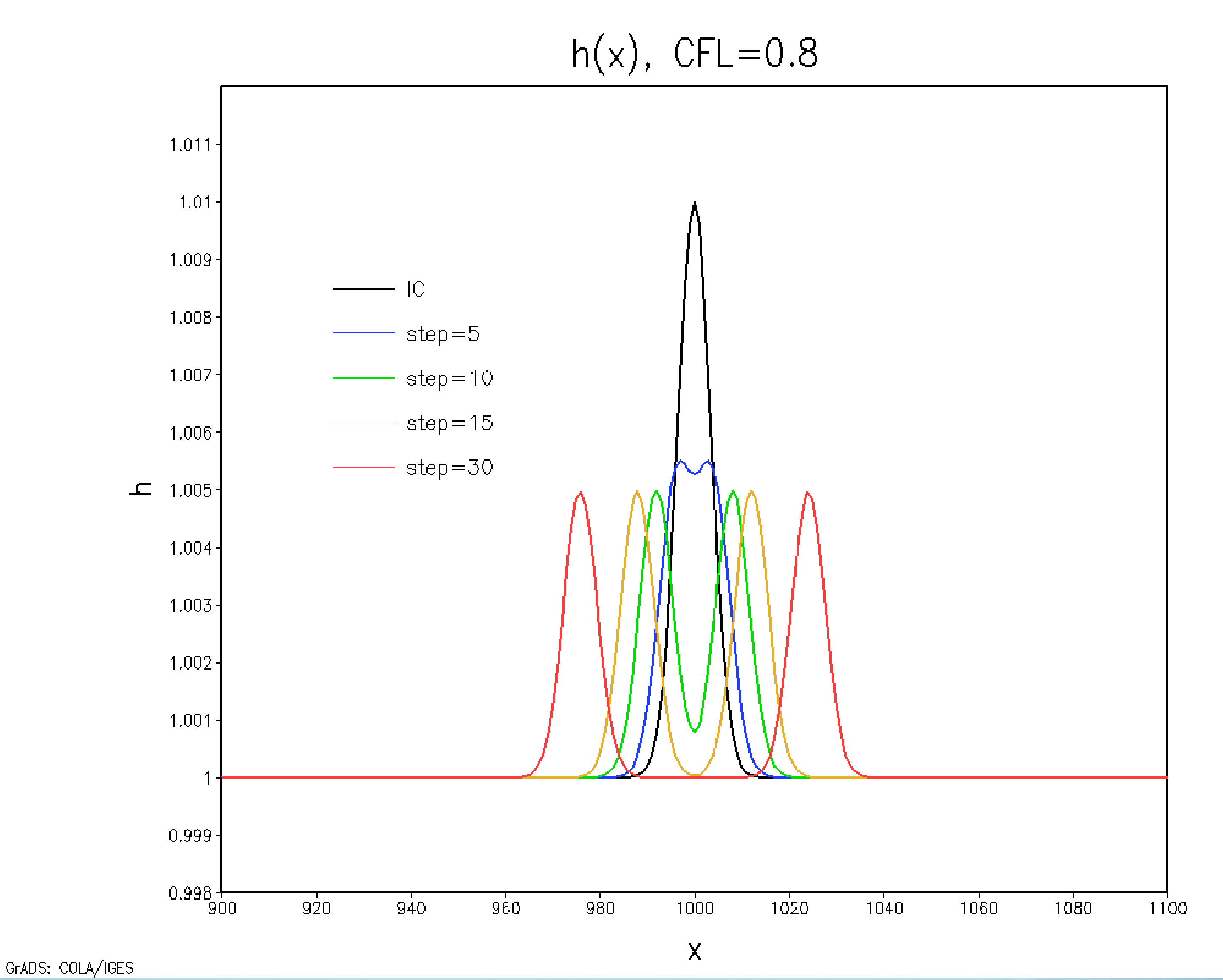
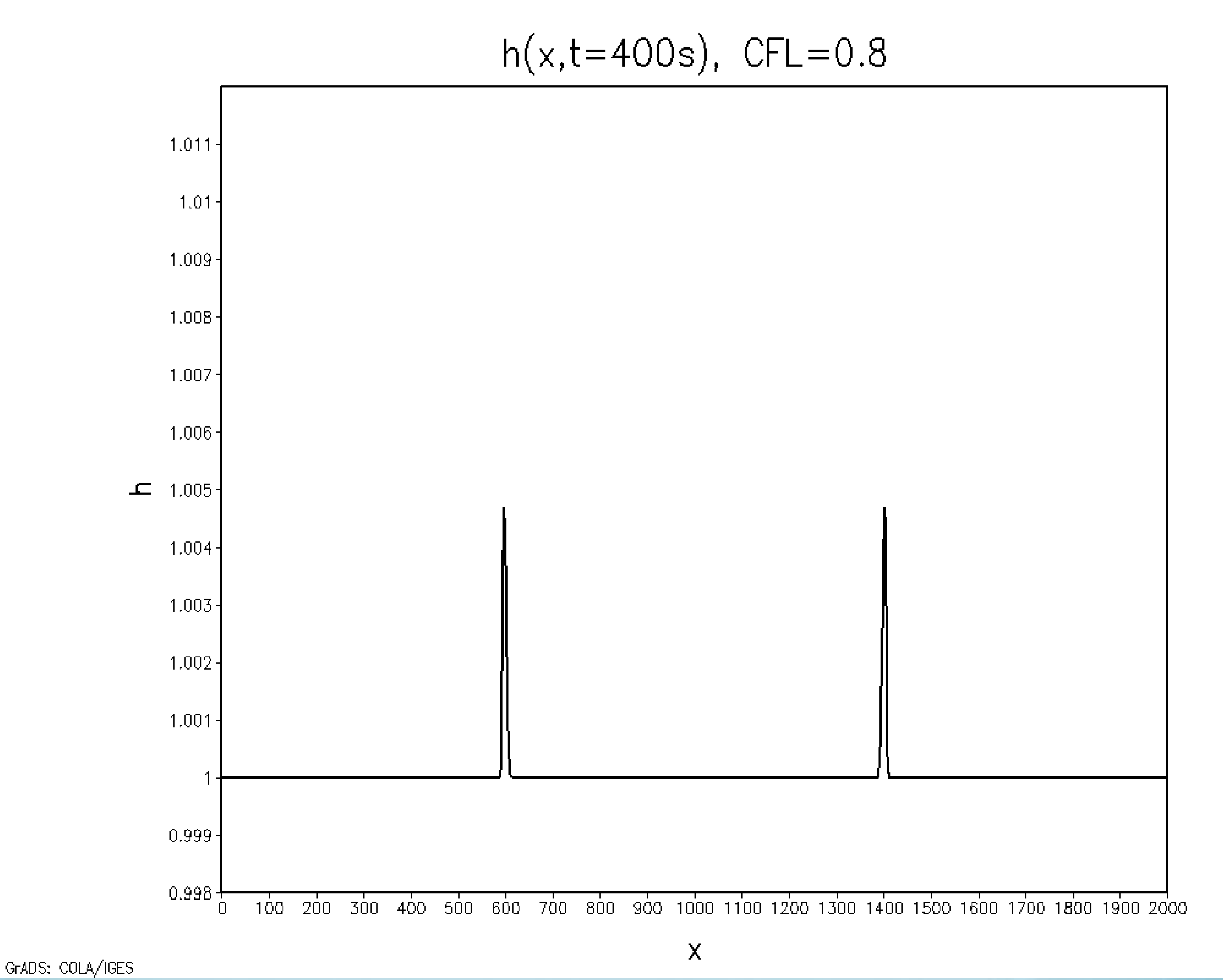
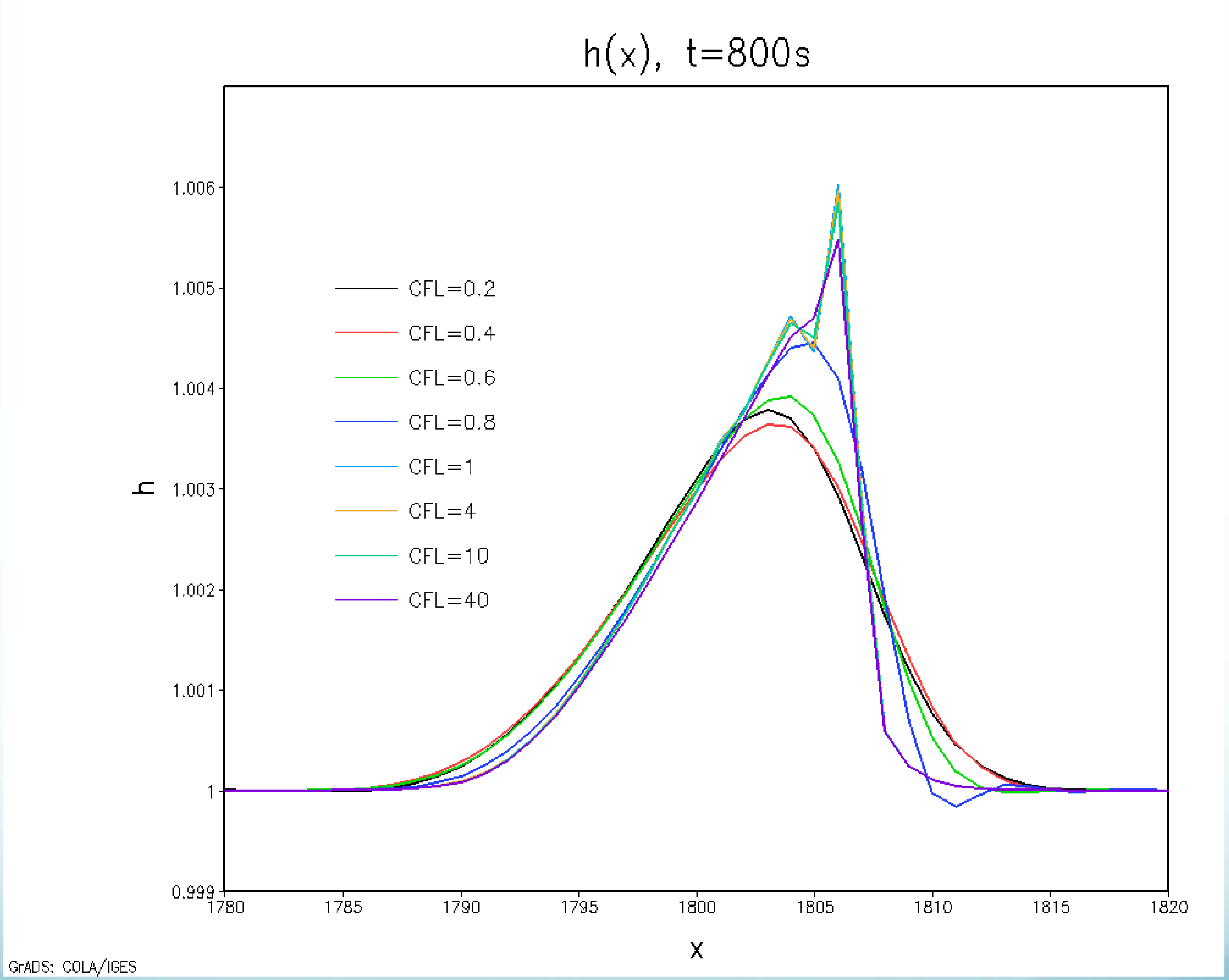


Fig. 13. The height at $t = 800.0$ under the initial condition of Eq. (39) for CFL = 0.4, 4.0, 10.0, 40.0.


h(x,t=0)
h
1.011
1.01
1.009
1.008
1.007
1.006
1.005
1.004
1.003
1.002
1.001
1
0.999
0.998
100 200 300 400 500 600 700 800 900 1000 1100 1200 1300 1400 1500 1600 1700 1800 1900 2000
x


h(x), CFL=0.8
1.011
1.01
1.009
1.008
1.007
1.006
1.005
1.004
1.003
1.002
1.001
1
0.999
0.998
h
IC
step=5
step=10
step=15
step=30
900
920
940
960
980
1000
1020
1040
1060
1080
1100
x
GrADS: COLA/IGES


h(x,t=400s), CFL=0.8
1.011
1.01
1.009
1.008
1.007
1.006
1.005
1.004
1.003
1.002
1.001
1
0.999
0.998
h
0 100 200 300 400 500 600 700 800 900 1000 1100 1200 1300 1400 1500 1600 1700 1800 1900 2000
X
GrADS: COLA/IGES



h(x), t=800s
CFL=0.2
CFL=0.4
CFL=0.6
CFL=0.8
CFL=1
CFL=4
CFL=10
CFL=40
h
1.006
1.005
1.004
1.003
1.002
1.001
1
0.999
1780
1785
1790
1795
1800
1805
1810
1815
1820
x
GrADS: COLA/IGES

advantages

- Advection and high frequency mode are merged into one prognostic variable
- Solve it with a pure semi-Lagrangian advection (here by NDSL)
- No need to select method to get gradient either by grid-point method or spectral method
- Put high frequency mode into the same time step as semi-Lagrangian scheme
- Much less diffusive than Eulerian scheme (such as results from Toda et al)

Nonhydrostatic system

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - R\overline{T}\frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial t} = -u\frac{\partial Q}{\partial x} - \gamma\left(\frac{\partial u}{\partial x}\right)$$

For Riemann solver, we let the above be

$$\frac{\partial}{\partial t}\begin{pmatrix} Q \\ u \end{pmatrix} + \begin{pmatrix} u & \gamma \\ R\overline{T} & u \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} Q \\ u \end{pmatrix} = 0$$

$$\frac{\partial R_1}{\partial t} = -c_1\frac{\partial R_1}{\partial x}$$

$$\frac{\partial R_2}{\partial t} = -c_2\frac{\partial R_2}{\partial x}$$

or

$$\frac{dR_1}{dt} = 0$$

$$\frac{dR_2}{dt} = 0$$

where

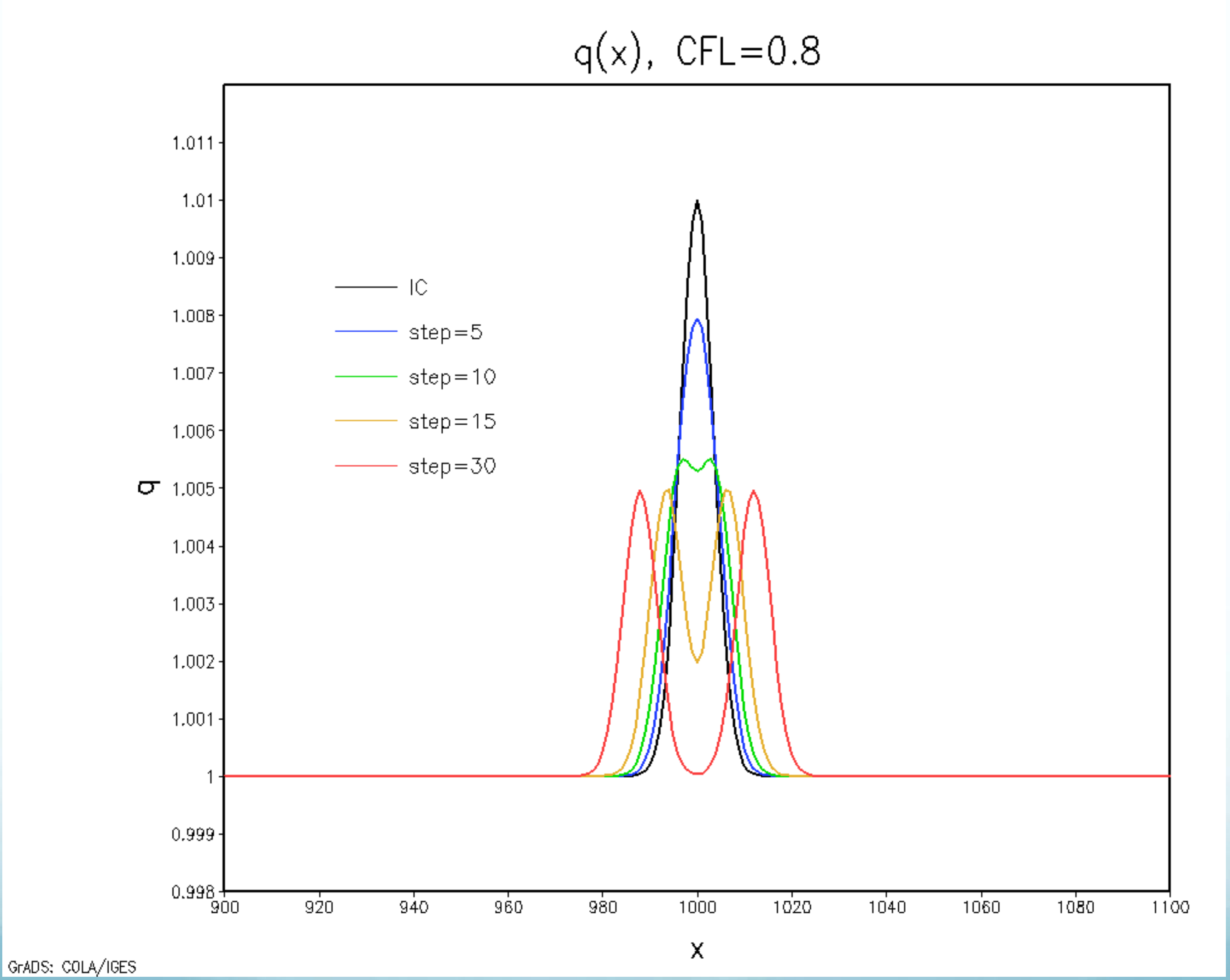
$$R_1 = \sqrt{R\overline{T}/\gamma}Q + u$$

$$R_2 = \sqrt{R\overline{T}/\gamma}Q - u$$

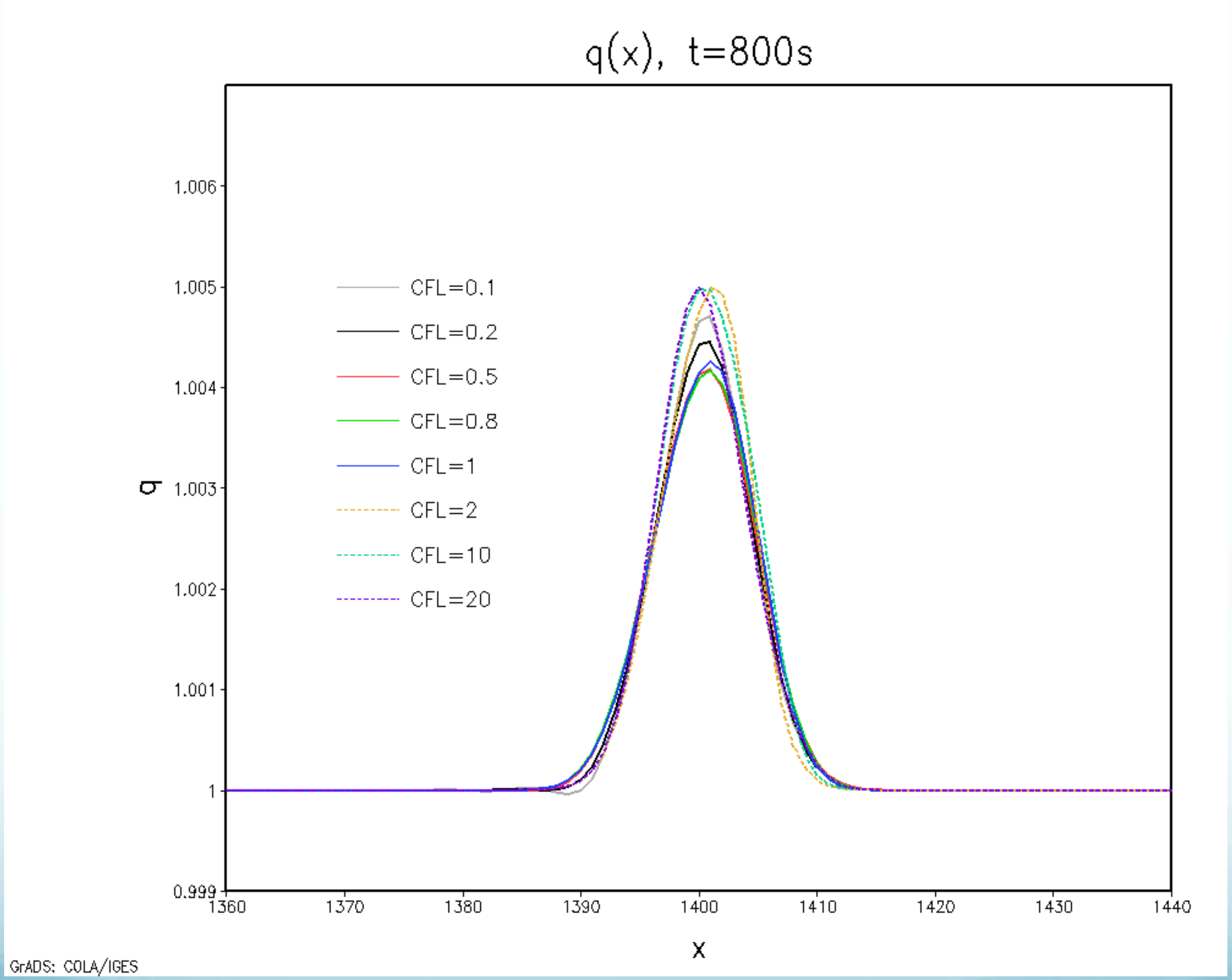
$$c_1 = u + \sqrt{\gamma R\overline{T}}$$

$$c_2 = u - \sqrt{\gamma R\overline{T}}$$

The initial acoustic spread



After 800s with different CFL



2D SWE on spherical coordinates can be written as

$$\frac{\partial u}{\partial t} + u\frac{1}{a\cos\phi}\frac{\partial u}{\partial \lambda} + v\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{g}{a\cos\phi}\frac{\partial H}{\partial \lambda} - \left(f + \frac{u\tan\phi}{a}\right)v = 0$$

$$\frac{\partial v}{\partial t} + u\frac{1}{a\cos\phi}\frac{\partial v}{\partial \lambda} + v\frac{1}{a}\frac{\partial v}{\partial \phi} + \frac{g}{a}\frac{\partial H}{\partial \phi} + \left(f + \frac{u\tan\phi}{a}\right)u = 0$$

$$\frac{\partial h}{\partial t} + u\frac{1}{a\cos\phi}\frac{\partial h}{\partial \lambda} + v\frac{1}{a}\frac{\partial h}{\partial \phi} + \frac{h}{a\cos\phi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial v\cos\phi}{\partial \phi}\right) = 0$$

where

$$u = a\cos\phi\frac{d\lambda}{dt}$$

$$v = a\frac{d\phi}{dt}$$

$$H = h + h_s$$

let

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \lambda'}$$

$$\frac{1}{a}\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \phi'}$$

rewrite the previous SWE into

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix} \frac{\partial}{\partial \lambda'}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \begin{pmatrix} v & 0 & h \\ 0 & v & 0 \\ g & 0 & v \end{pmatrix} \frac{\partial}{\partial \phi'}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \begin{pmatrix} -\frac{\tan\phi}{a} hv \\ -fv - \frac{\tan\phi}{a} uv + g\frac{\partial h_s}{\partial \lambda'} \\ fu + \frac{\tan\phi}{a} uu + g\frac{\partial h_s}{\partial \phi'} \end{pmatrix} = 0$$

Then dimensional split into

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix} \frac{\partial}{\partial \lambda'}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -fv - \frac{\tan\phi}{a} uv + g\frac{\partial h_s}{\partial \lambda'} \\ fu + \frac{\tan\phi}{a} uu + g\frac{\partial h_s}{\partial \phi'} \end{pmatrix} = 0$$

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \begin{pmatrix} v & 0 & h \\ 0 & v & 0 \\ g & 0 & v \end{pmatrix} \frac{\partial}{\partial \phi'}\begin{pmatrix} h \\ u \\ v \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2\frac{\tan\phi}{a} hv \\ -fv - \frac{\tan\phi}{a} uv + g\frac{\partial h_s}{\partial \lambda'} \\ fu + \frac{\tan\phi}{a} uu + g\frac{\partial h_s}{\partial \phi'} \end{pmatrix} = 0$$

From latitudinal direction, we have

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ g & u \end{pmatrix}\frac{\partial}{\partial \lambda'}\begin{pmatrix} h \\ u \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ \left(f + \frac{\tan\phi}{a}u\right)v \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ g\frac{\partial h_s}{\partial \lambda'} \end{pmatrix} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \lambda'} + \frac{1}{2}\left(f + \frac{\tan\phi}{a}u\right)u + \frac{1}{2}\left(g\frac{\partial h_s}{\partial \phi'}\right) = 0$$

So we need matrix L^{-1} and L to satisfy following

$$L^{-1}\begin{pmatrix} u & h \\ g & u \end{pmatrix}L = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \quad \& \quad LL^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to diagonalize to get eigenvalues by

$$\begin{pmatrix} u & h \\ g & u \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{so} \quad \left\langle \begin{pmatrix} u & h \\ g & u \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right\rangle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$Det\begin{vmatrix} u-C & h \\ g & u-C \end{vmatrix} = 0 \quad \text{so} \quad \begin{aligned} C_1 &= u + \sqrt{gh} \\ C_2 &= u - \sqrt{gh} \end{aligned}$$

In summary, we have three equations in advection form as

$$\frac{\partial R_1}{\partial t} + C_1 \frac{\partial R_1}{\partial \lambda'} - \frac{1}{4}\left[f + \frac{\tan\phi}{a} u \right] v + \frac{1}{4}\frac{\partial g h_s}{\partial \lambda'} = 0$$

$$\frac{\partial R_2}{\partial t} + C_2 \frac{\partial R_2}{\partial \lambda'} + \frac{1}{4}\left[f + \frac{\tan\phi}{a} u \right] v - \frac{1}{4}\frac{\partial g h_s}{\partial \lambda'} = 0$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial \lambda'} + \frac{1}{2}\left(f + \frac{\tan\phi}{a} u \right) u + \frac{1}{2} g \frac{\partial h_s}{\partial \varphi'} = 0$$

$$R_1 = \sqrt{gh} + u/2$$

$$R_2 = \sqrt{gh} - u/2$$

$$C_1 = u + \sqrt{gh}$$

$$C_2 = u - \sqrt{gh}$$

then we solve R and v by C and u with semi-Lagrangain

$$R_1^a = R_1^d + \frac{\Delta t}{4} f_m v^m - \frac{\Delta t}{4}\left(\frac{(gh_s)^a - (gh_s)^d}{(\lambda'^a - \lambda'^d)} \right)$$

$$R_2^a = R_2^d - \frac{\Delta t}{4} f_m v^m + \frac{\Delta t}{4}\left(\frac{(gh_s)^a - (gh_s)^d}{(\lambda'^a - \lambda'^d)} \right)$$

$$v^a = v^d - \frac{\Delta t}{2} f_m u^m - \frac{\Delta t}{4}\left\langle \left(g \frac{\partial h_s}{\partial \phi'} \right)^a + \left(g \frac{\partial h_s}{\partial \phi'} \right)^d \right\rangle$$

$$f_m = \left(f + \frac{\tan\phi}{a} u \right)^m$$

$$h^{t+\Delta t} = \frac{1}{g}\left(\frac{R_1^{t+\Delta t} + R_2^{t+\Delta t}}{2} \right)^2$$

$$u^{t+\Delta t} = R_1^{t+\Delta t} - R_2^{t+\Delta t}$$

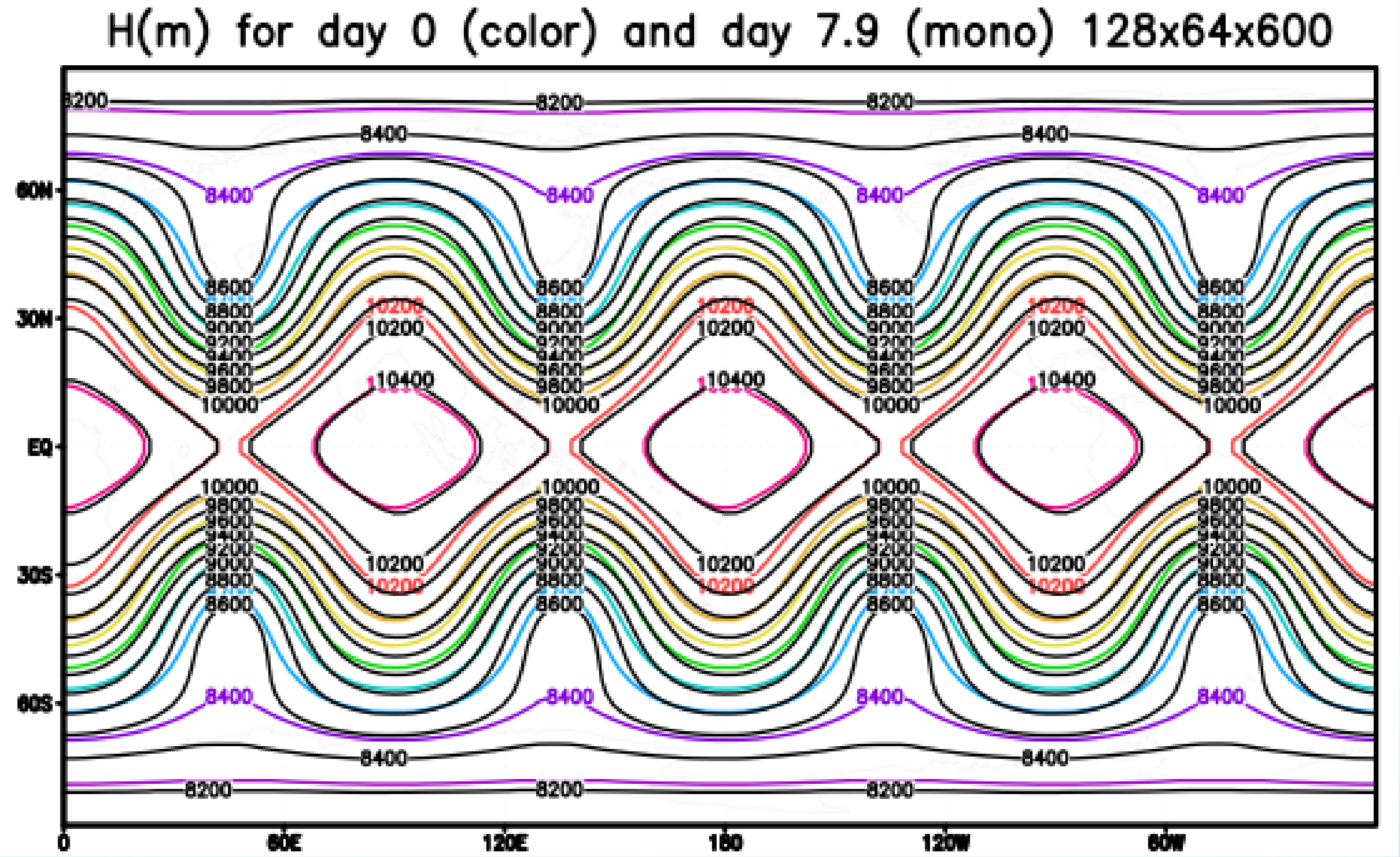
The same procedure in latitudinal direction.

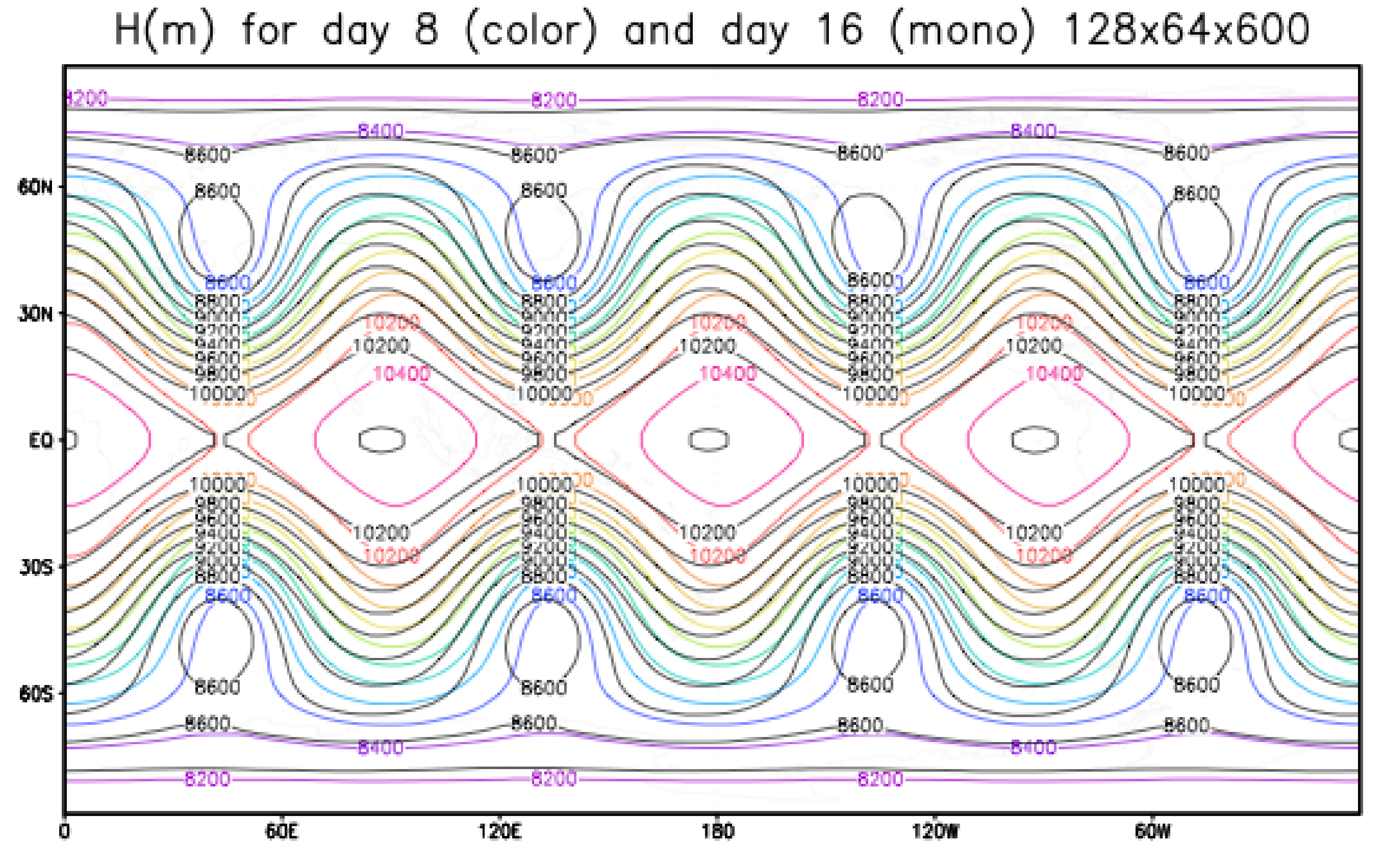
Then

The wavenumber 4 Rossby-Haurwitz wave on sphere as case 6 of Williamson et al 1992 is tested.

The non-iteration dimensional-split semi-Lagrangian is used for solving the equation as mentioned.
Reduced Gaussian grid is carried as model grid.
Equal latitide/longitude A grid is utilized for splitting.

Cubic Spline under tension is used for interpolation.
No mass conservation is considered yet.
No polar filter is applied yet.


H(m) for day 0 (color) and day 7.9 (mono) 128x64x600
8200
8400
8600
8800
9000
9200
9400
9600
9800
10000
10200
10400
60N
30N
EQ
30S
60S
0
60E
120E
180
120W
60W


H(m) for day 8 (color) and day 16 (mono) 128x64x600
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30N
EQ
30S
60S
0
60E
120E
180
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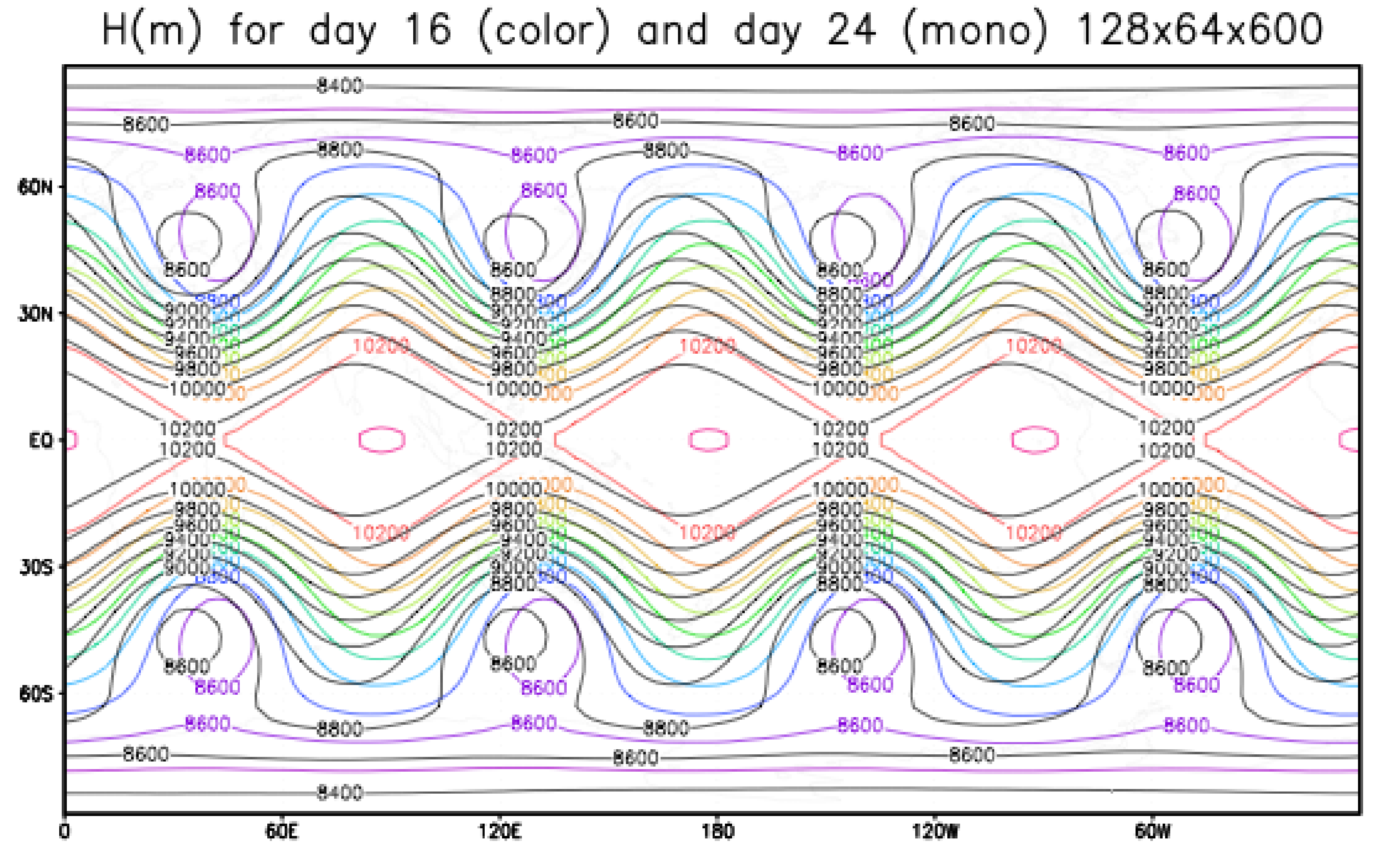
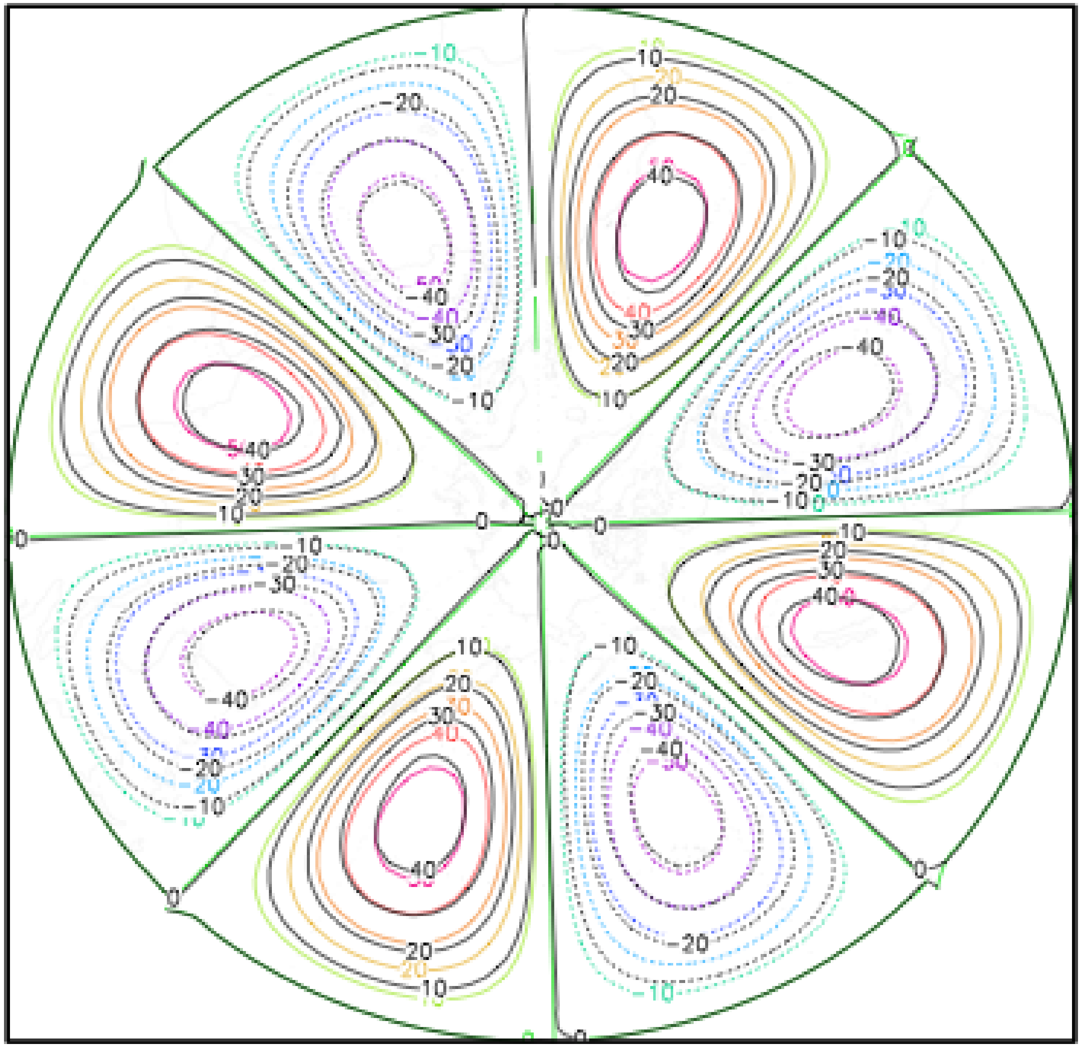
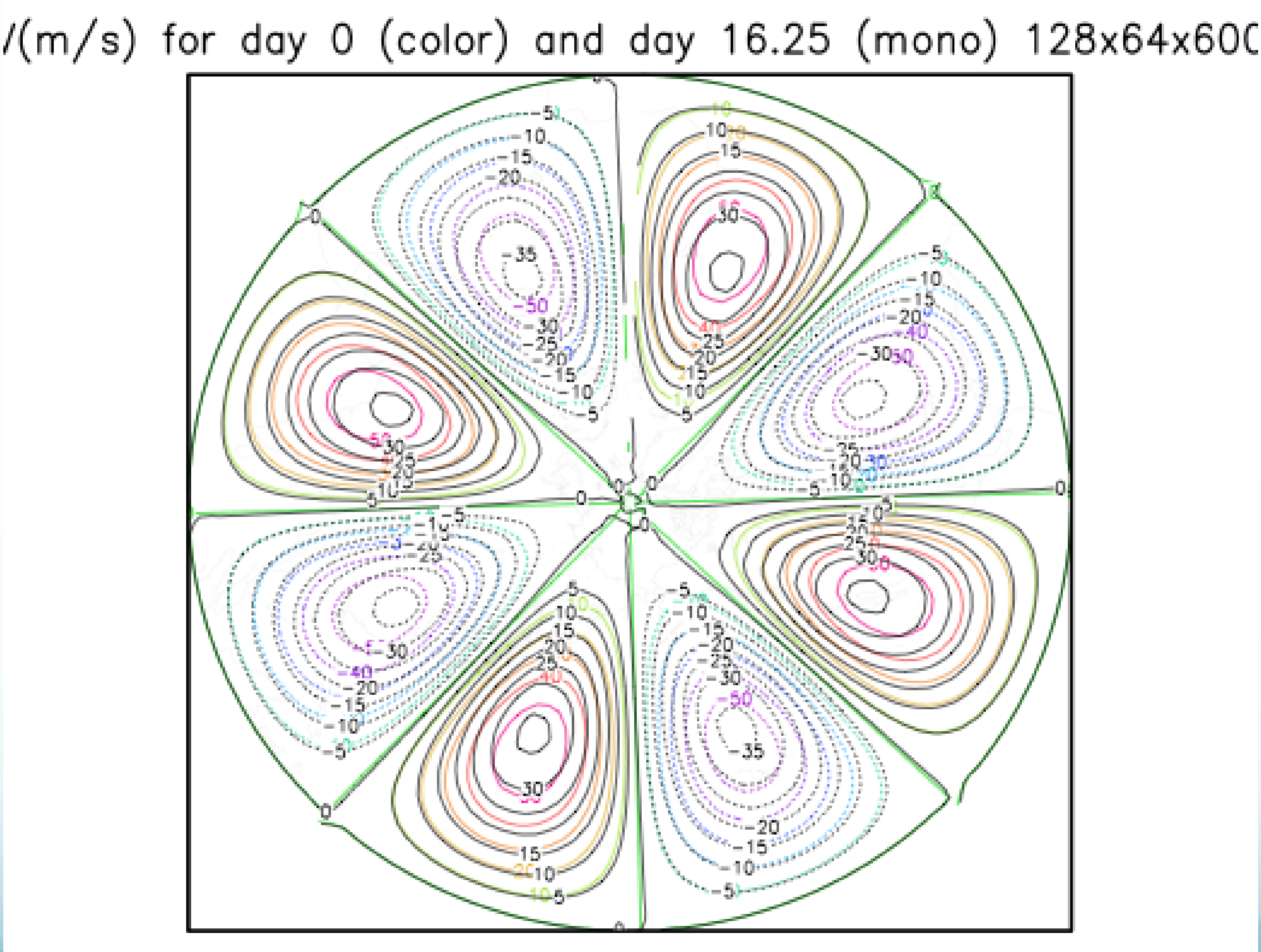
9800
10000
10200
10400

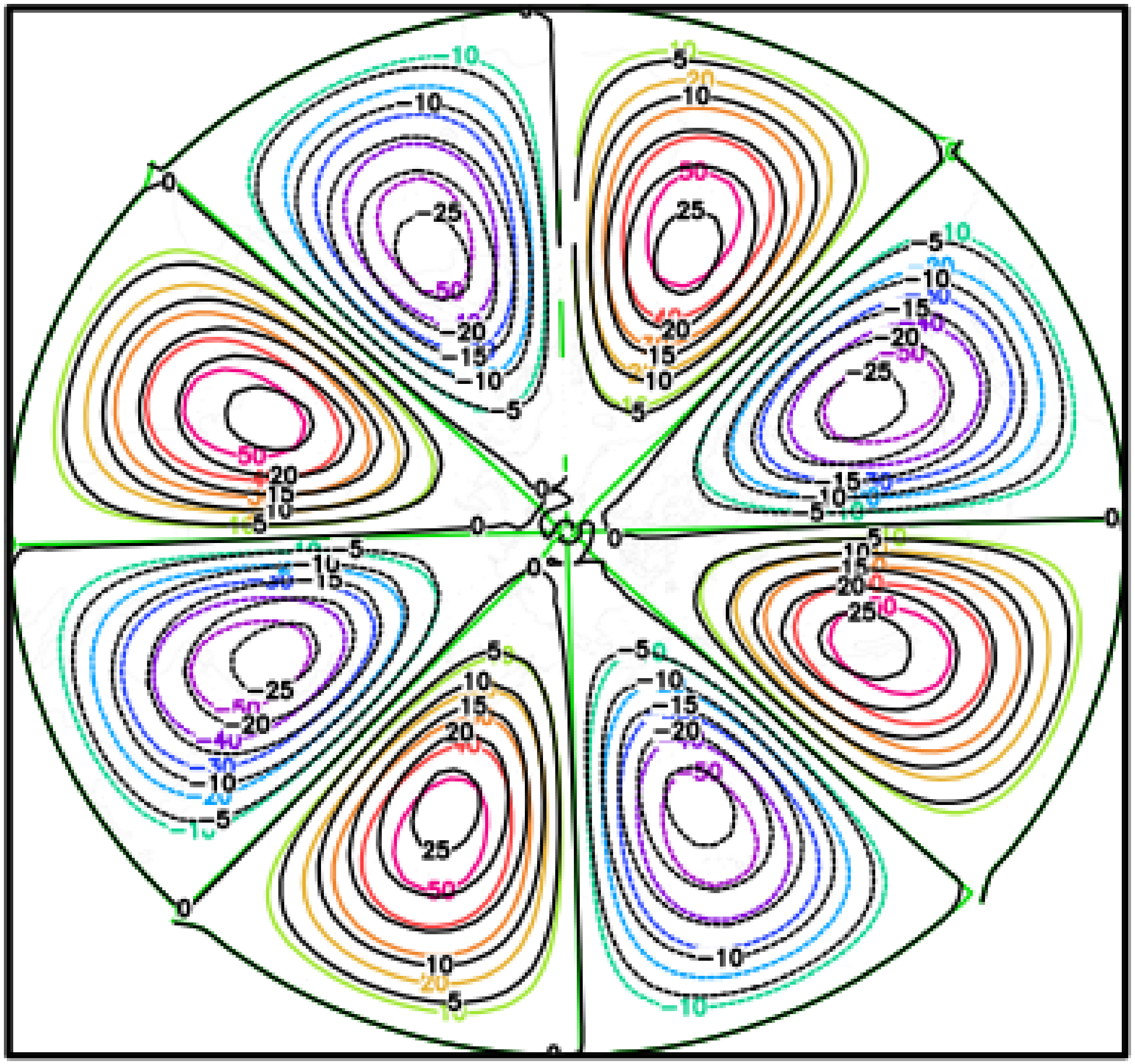

H(m) for day 16 (color) and day 24 (mono) 128x64x600
60N
30N
EQ
30S
60S
0
60E
120E
180
120W
60W

V(m/s) for day 0 (color) and day 7.9 (mono) 128x64x600




/(m/s) for day 0 (color) and day 16.25 (mono) 128x64x60C

V(m/s) for day 0 (color) and day 25 (mono) 128x64x600



2D nonhydrostatic tests in x-z with isotherm

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - R\overline{T}\frac{\partial Q'}{\partial x}$$

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - R\overline{T}\frac{\partial Q'}{\partial z}$$

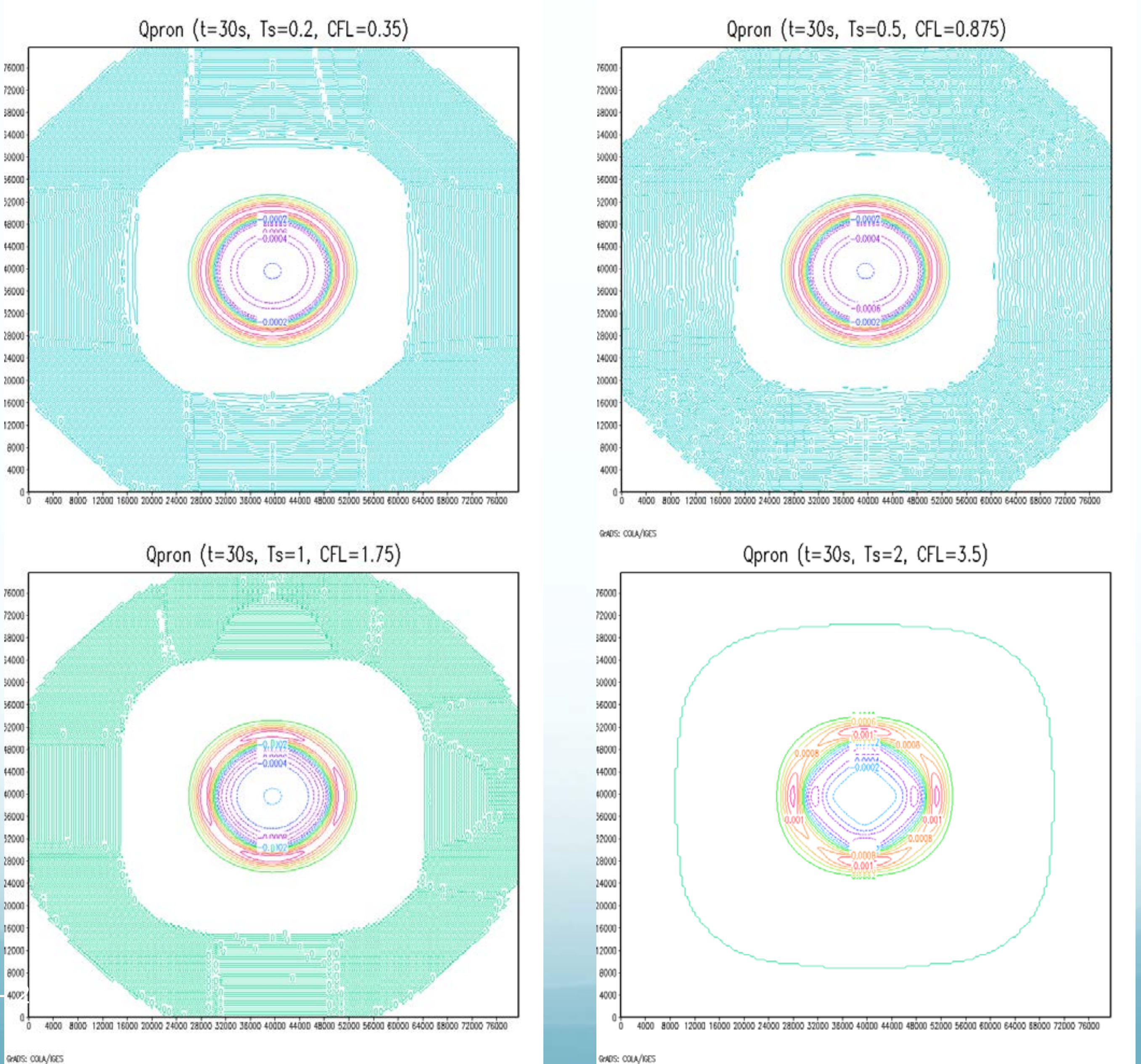
$$\frac{\partial Q'}{\partial t} = -u\frac{\partial Q'}{\partial x} - w\frac{\partial Q'}{\partial z} - \gamma\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{gw}{R\overline{T}}$$

where

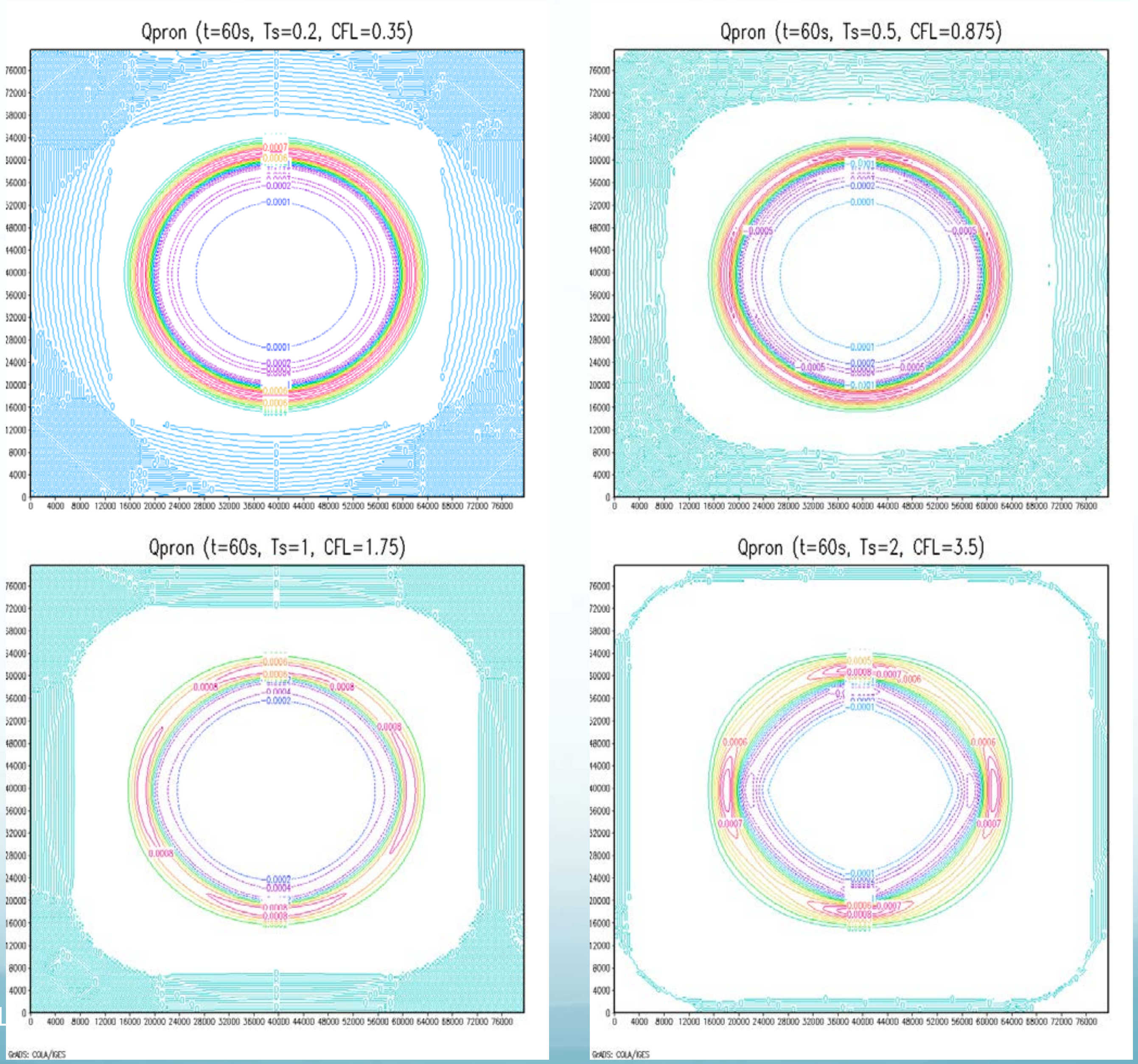
$$\frac{\partial \overline{Q}}{\partial z} = -\frac{g}{R\overline{T}}$$

$$Q = \overline{Q} + Q'$$

2D tests in x-z (non-forcing) – Q’ (t=30s)



2D tests in x-z (non-forcing) – Q’ (t=60s)



Discussion

- 1D system is impeccable.
- 2D system should be evaluated carefully
  - CFL >1 shows distortion
  - Splitting error?
- While 2D has no problem, 3D system can be done by applying splitting method, do 2D in horizontal first then 1D in vertical.
- Since it becomes an advection equation, semi-Lagrangian with splitting can be used to solve it with stable integration.